User Equilibrium Model of Ridesharing Transport with High-Occupancy Vehicles Lane

Phathinan THAITHATKUL · Toru SEO · Takahiko KUSAKABE · Yasuo ASAKURA

Abstract This study formulates the traffic equilibrium problem for ridesharing with the availability of high-occupancy vehicles lane in future transport system, where the mean of transport for personal trip is expected to be shared autonomous vehicles. For the sake of mathematical analysis, the problem is also formulated in a variational inequalities form, and the existence and uniqueness of user equilibrium flow pattern are mathematically analyzed accordingly.

Keywords: Ridesharing transport · User equilibrium model · Variational inequalities problem · Modal split

1 Introduction

Ridesharing transport has been introduced as one of the transport services that have the potential to fulfil the conventional public transport system for the areas in which demand is low and spread over (Mulley and Nelson 2009). Ridesharing can provide
an on-demand and door-to-door service as a personal car, while maintaining the ability to reduce congestion and emissions as a public transport (Nielsen et al. 2015). It has been broadly studied to efficiently and effectively operate the ridesharing system, especially in its matching problem (Agatz et al. 2011; Kleiner et al. 2011; Thaithatkul et al. 2015). However, ridesharing studies on traffic assignment problem (TAP) aspect are still limited.

With the availability of ridesharing, there is an interaction among ridesharing and other modes of transport on the same link. In general, travel cost is not the same among different modes of transport. This means that the problem with ridesharing can be considered as a TAP with asymmetric link interactions. Dafermos (1980) proposed an alternative formulation for TAP with asymmetric link interactions using techniques of variational inequalities theory. The proof of existence and uniqueness of user equilibrium (UE) was also provided in his study. Moreover, its UE flow pattern can be found by the existing algorithm for standard TAP (Sheffi 1985).

According to the ridesharing-related literatures, Xu et al. (2015) developed a model for UE with ridesharing based on the variational inequalities theory. Their framework is considering a ridesharing that requires ridesharing driver who will get income from sharing his ride to ridesharing passenger(s) who will pay. Their model can be used to analyze the UE modal split (i.e., modal split among solo driver, ridesharing driver, ridesharing passenger) at different situations (e.g., increase of ridesharing price). Furthermore, Bahat and Bekhor (2016) also developed the TAP-based ridesharing UE model considering the same three choices of transport mode. Additionally, their model was incorporated with a discrete choice model, and could explicitly describe flow-dependent waiting time of ridesharing passenger for an available ridesharing driver.

From the present development of autonomous vehicles (Fagnant and Kockelman 2015) and sharing economy concept (Botsman and Roger 2010), it can be expected that travelers will travel by using shared autonomous vehicles for their private journey in future rather than using their own private vehicles. Moreover, traveler’s ridesharing behaviors are expected to be changed as the role of driver might be mitigated. For example, the pricing mechanism between ridesharing passengers and drivers as well as the ridesharing passenger’s hesitation might not exist. To encourage the use of ridesharing in the transport system with shared autonomous vehicles, the concept of high-occupancy vehicle (HOV) lane—where ridesharing travelers are eligible to use HOV lane to reduce their travel time and/or cost—is still applicable. However, with the change of traveler’s ridesharing behaviors, the use of HOV lane may also change. As a consequence, redesign of HOV lane (e.g., capacity, location) will also be needed in order to effectively and efficiently operate this transport system.

Therefore, the framework of this study is to consider a ridesharing using shared autonomous vehicle with the availability of HOV lane. The objectives of this study are: to formulate the ridesharing UE model that can represent the mutual dependence of in-vehicle travel time among travel modes for the considering framework, and to analytically analyze the UE’s characteristics (e.g., existence, uniqueness). To do so, the considered problem is specifically described in Section 2. A traffic equilibrium
problem for ridesharing is formulated in Section 3.1. As previously mentioned, problem with asymmetric link costs is difficult to mathematically solve and analyze for the UE and its characteristics, therefore, the problem is also formulated in a variational inequalities problem in Section 3.2. Based on this formulation, the existence and uniqueness of UE are mathematically analyzed in Section 4. The possible applications of the formulated model, its implications, and future research direction are discussed in Section 5.

2 Problem Description

We consider the future transport system where travelers do not own a private vehicle, but they rather make a personal trip by using the shared autonomous vehicles. Therefore, the role of driver is not considered. This framework can also be applied to the taxi-sharing system in present transport system. The availability of HOV lanes is also considered. As the first step, the ridesharing UE model in this framework is formulated based on following assumptions.

- Network with one origin-destination pair connected by one link.
- The link consists of two types of lane: HOV lane where only vehicles with more than one rider (ridesharing vehicles) are allowed, and regular lane.
- Lane changing behavior is not allowed.
- Vehicle is for-hire vehicle and its supply is always sufficient.
- Vehicles are homogeneous in terms of capacity.
- Travel mode choice is limited to riding alone and ridesharing.
- Travelers are homogeneous in terms of generalized cost function and value of time.
- Individual generalized cost for traveling alone consists of congestion cost and travel fare.
- Individual generalized cost for ridesharing consists of congestion cost, discounted travel fare, and discomfort cost (when sharing private space with others).
- The congestion cost is a monotonically increasing corresponding to the number of vehicles.
- A fare function and discomfort cost function are dependent only on travel time.

3 Formulation

The traffic equilibrium problem for ridesharing is formulated in Section 3.1. The graphical example for the existence of user equilibrium is also provided and explained. For the sake of mathematical analysis on the UE characteristics, the UE model for ridesharing is also formulated in a variational inequalities form in Section 3.2. The variables and parameters used in model formulation are listed and described in Table 1 unless otherwise specified.
### Table 1 Variable and parameters used in model formulation

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>Route (representing travel mode and lane choices in this paper)</td>
</tr>
<tr>
<td>( p_a )</td>
<td>A set of route choices</td>
</tr>
<tr>
<td>( a )</td>
<td>Link</td>
</tr>
<tr>
<td>( a_a )</td>
<td>A set of links in the cost-represented network</td>
</tr>
<tr>
<td>( N )</td>
<td>Total travel demand of travelers</td>
</tr>
<tr>
<td>( n_p )</td>
<td>Number of travelers on route ( p )</td>
</tr>
<tr>
<td>( n )</td>
<td>A vector of number of travelers</td>
</tr>
<tr>
<td>( F_p(n_p) )</td>
<td>Number of vehicles on route ( p ) which depends on number of travelers on that route, i.e., route flow</td>
</tr>
<tr>
<td>( F(a) )</td>
<td>A vector of route flow, i.e., route flow pattern</td>
</tr>
<tr>
<td>( x )</td>
<td>Number of vehicles</td>
</tr>
<tr>
<td>( x_a )</td>
<td>Number of vehicles on link ( a ), i.e., link flow</td>
</tr>
<tr>
<td>( x_x )</td>
<td>A vector of link flow, i.e., link flow pattern</td>
</tr>
<tr>
<td>( X )</td>
<td>A set of all feasible link flow patterns</td>
</tr>
<tr>
<td>( t(x) )</td>
<td>A function of travel time which depends on number of vehicles</td>
</tr>
<tr>
<td>( t_a(x) )</td>
<td>A function of travel time of link ( a )</td>
</tr>
<tr>
<td>( Z^A )</td>
<td>An individual generalized cost for traveling alone and ridesharing, respectively.</td>
</tr>
<tr>
<td>( Z_a )</td>
<td>A route cost function, i.e., an individual generalized cost of route ( p )</td>
</tr>
<tr>
<td>( Z )</td>
<td>A vector of route cost functions</td>
</tr>
<tr>
<td>( z_a )</td>
<td>A link cost function, i.e., an individual generalized cost of link ( a )</td>
</tr>
<tr>
<td>( z )</td>
<td>A vector of link cost functions</td>
</tr>
<tr>
<td>( f(T(x)) )</td>
<td>A travel fare function which depends on travel time</td>
</tr>
<tr>
<td>( g(T(x)) )</td>
<td>A discomfort cost function which depends on travel time</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>A value of travel time</td>
</tr>
<tr>
<td>( r )</td>
<td>A discount rate of travel fare for ridesharing</td>
</tr>
<tr>
<td>( c )</td>
<td>A capacity of seats in a vehicle</td>
</tr>
</tbody>
</table>

3.1 Formulation of traffic equilibrium problem

An individual generalized cost for traveling alone \( Z^A \) is defined as

\[
Z^A(T(x)) = \sigma T(x) + f(T(x)),
\]  

(1)

consisting of a cost of travel time and a travel fare. For travelers who rideshare, the travel fare can be discounted at rate \( r \). However, there will be a discomfort cost occurred because of the sharing of private space. Therefore, an individual generalized cost for ridesharing \( Z^{RS} \) is defined as

\[
Z^{RS}(T(x)) = \sigma T(x) + f(T(x)) - rf(T(x)) + g(T(x)).
\]  

(2)

To formulate this problem to conform with the standard traffic equilibrium problem, the original physical network (Figure 1(a)) is represented by a new network, namely cost-represented network (Figure 1(b)). The link cost functions \( z_a \), where \( a \in a \) and \( a = \{1,2,2'\} \), are defined as
Link 1 represents the physical HOV lane. The cost of link 1, $z_1$, is equivalent to a cost of ridesharing (Equation 3) which is a function of its flow (i.e., number of vehicles $x_1$), as there are only ridesharing travelers allowed on HOV lane. Link 2 represents the physical regular lane. As any travelers can use this lane, its flow $x_2$ may include both traveling alone and ridesharing travelers; however, the costs of riding alone and ridesharing on regular lane are different. To represent this situation, the cost of link 2 is represented by a cost of traveling alone (Equation 4). The dummy link $2'$ (dashed line in Figure 1(b)) represents the additional cost for ridesharing on regular lane (Equation 5), which is a function of flow of link 2.

Based on this cost-represented network, the choices of travel mode and lane are considered as a choice of route $p$, where $p \in \mathbf{p}$. Therefore, there are three choices of route ($\mathbf{p} = \{1,2,3\}$): ridesharing on HOV lane ($p = 1$), traveling alone on regular lane ($p = 2$), and ridesharing on regular lane ($p = 3$). The link-route incidence matrix $\delta_{ap}$ is defined as

$$
\delta_{ap} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
$$

With this link-route incidence matrix, the route cost $Z_p$ can be obtained by

$$
Z_p = \sum_a \delta_{ap} g e_a
$$

conforming with Equations (1) and (2). This means that, to rideshare on regular lane, travelers will experience both costs of link 2 and link $2'$. The route flow $F_p$ is a function of number of travelers on that route $n_p$, which can be expressed as

$$
F_p(n_p) = n_p/c.
$$
\[ F_2(n_2) = n_2, \]  
\[ F_3(n_3) = n_3/c, \]  
with the demand constraint
\[ \sum_p n_p = N. \]

where \( N \) denotes a given total travel demand and \( c \) denotes the capacity of a vehicle. Once we know the number of travelers \( n_p \), the link flow \( x_a \) can then be obtained as
\[ x_a = \sum_p \delta_{ap} F_p (n_p). \]

Note that, as link 2′ is a dummy link for calculating the additional cost of ridesharing on regular lane, its flow \( x_{2′} \) is also dummy and does not exist in physical network. According to the definition of UE (Wardrop 1952), the flow pattern at UE is flow pattern where no one has incentive of unilaterally changing decisions (i.e., routes). In other words, the UE is realized when generalized costs for all used routes are less than or equal to those of unused routes. Given a set of number of travelers on each route at the UE as \( \mathbf{n}^∗ \), \( \mathbf{n}^∗ \) must satisfy
\[ Z_p(F(n^∗)) > Z_q(F(n^∗)) \text{ implies } F_p(n^∗) = 0 \text{ or } \mathbf{n}^∗ = 0 \]  
\[ (13) \]

where \( F(\cdot) \) is a vector of route flows (i.e., route flow pattern). Since the routes with lowest cost are only used, the total cost cannot be further reduced by changing routes. The UE condition based on route formulation can be written as
\[ \mathbf{z}(F(n^∗)) \cdot (F(n) - F(n^∗)) \geq 0 \]  
\[ (14) \]

where \( \mathbf{z}(\cdot) \) denotes a vector of route costs. With Equations (7) and (12), the total route costs and total link costs are equivalent.
\[ \mathbf{z}(F(n^∗)) \cdot F(n) = \mathbf{z}(\mathbf{x}^∗) \cdot x \]  
\[ (15) \]

where \( \mathbf{z} \) is a vector of link costs, \( x \) is vector of link flows (i.e., link flow pattern), and \( \mathbf{x}^∗ \) is a vector of link flows at UE. Therefore, for the sake of computation, the UE condition can also be expressed as a link formulation,
\[ \mathbf{z}(\mathbf{x}^∗) \cdot (x - \mathbf{x}^∗) \geq 0. \]  
\[ (16) \]

**Graphical example of existence of user equilibrium**

For a better understanding of existence of UE based on this formulation, we consider following simple scenario. The HOV lane is assumed to have better condition but
lower capacity comparing to the regular lane. The travel time function of HOV and regular lanes are shown in Figure 2. The travel fare $f(T_a(x_a))$ is linearly increasing corresponding to travel time (Figure 3). Assuming that the fare can be discounted at the fixed rate $r$ for ridesharing, so that the function of fare reduction $rf(T_a(x_a))$ is also linear. Considering the situation where travelers feel uncomfortable to rideshare, once they start ridesharing trip, some amount of discomfort cost $g(T_a(x_a))$ occurs and linearly increases over travel time (Figure 4). Given the value of time $\sigma$ a positive constant value, with this scenario, the generalized cost for each travel route $Z_{\sigma}$ can be illustrated as shown in Figure 5.
With the above specified scenario, the UE and its link flow pattern (i.e., modal split) at different travel demand level can be realized through graphical representation as shown in Figure 6. The upper graph in Figure 6 is consistent with Figure 2 where the axes are swapped. The lower one is consistent with Figure 5. It shows that at the different total travel demand \( N \), there can be four different kinds of UE as following explanation.
For total travel demand $N$ is less than $x^*_2$, the following UE is realized. At this UE, all travelers travel alone in regular lane as the generalized cost for traveling alone is always less than that of ridesharing on both HOV and regular lanes.

For total travel demand $N$ is between $(x^*_2, cx^*_1 + x^*_2]$, it exists of UE where there is always one proportion between ridesharing in HOV lane and traveling alone in regular lane that makes these two routes cost the same (i.e., $Z_1 = Z_2$). Note that there is no traveler ridesharing in regular lane because it will never cost less than traveling alone in regular lane with this range of possible travel time. If the modal split is not at this proportion, the mode that has larger number of travelers than such proportion will cost higher than another mode due to the increase of travel time, so that travelers will change their travel mode back, and such proportion will be eventually reached.

For total travel demand $N$ is between $(cx^*_1 + x^*_2, cx^*_1 + cx^*_2)$, there is an existence of UE at the intercept point of generalized cost functions for ridesharing and traveling alone. At this UE, all travel modes and lanes are used as they cost the same where there are $x^*_1$ and $x^*_2$ vehicles in HOV and regular lanes, respectively. This UE is considered stable because if the proportion of vehicles changes, it will eventually converge back to this proportion. For example, travelers who travel alone in regular lane change their mode to be ridesharing in regular lane (i.e., $x^*_2 < x^*_2$); even though the travel time in regular lane is reduced because of a lower number of vehicles in the regular lane, ridesharing at that travel time costs more than traveling alone due to the additional cost for ridesharing is larger than the fare reduction. This means that traveler will change travel mode back to traveling alone in regular lane.

Once the $N$ is larger than or equal to $cx^*_1 + cx^*_2$, the realized UE is where all travelers rideshare using both HOV and regular lanes. Moreover, at the UE where $Z_1 = Z_3$, there are always more travelers using regular lane comparing to HOV lane. This UE is also stable as, for example, if travelers change from regular lane to HOV lane, the HOV lane will become congested with longer travel time meaning that using HOV lane will cost more than using regular lane. Moreover, if $N$ is as large as mentioned, travelers can always enjoy the benefit of fare reduction from ridesharing. In other words, the fare reduction amount is larger than the discomfort cost.

### 3.2 Formulation of traffic equilibrium problem in variational inequalities form

According to the model proposed by Dafermos (1980), in order to have a cost function with symmetric positive definite of its Jacobian matrix, a new link cost function $\bar{z}(x)$ can be introduced as follows

$$\bar{z}(x) = Mx + h,$$  \hspace{1cm} (17)

where

---

1 The traffic equilibrium problem with the symmetric Jacobian matrix of its cost functions can be easily proven for the uniqueness of its UE solution (i.e., Jacobian matrix has positive definite). That is UE solution is unique if all eigenvalues of symmetric Jacobian matrix are positive.
\[ h = \rho z(x) - Mx, \]  

(18)

\( M \) is a symmetric positive definite matrix, \( \rho \) is positive number, and \( \bar{x} \) is some fixed possible link flow pattern.

Let \( x^* \) be a link flow pattern at UE, \( x^* \) must satisfy

\[ \bar{z}(x^*) \cdot (x - x^*) \geq 0, \quad \text{for } \forall x \in X \]  

(19)

or

\[ x^* \cdot M(x - x^*) + h \cdot (x - x^*) \geq 0, \quad \text{for } \forall x \in X \]  

(20)

where \( X \) denotes a set of all feasible link flow patterns that satisfy Equations (11) and (12) at a given total travel demand \( N \). The UE condition in Equation (20) is equivalent to the UE condition of original cost function in Equation (16) by substituting Equation (18) into Equation (20). That is, given that \( \bar{x} \) in Equation (18) satisfies UE condition in Equation (20) such that \( \bar{x} = x^* \),

\[ x^* \cdot M(x - x^*) + \rho z(x^*) \cdot (x - x^*) \geq 0, \]

\[ x^* \cdot M(x - x^*) + \rho z(x^*) \cdot (x - x^*) - Mx^* \cdot (x - x^*) \geq 0, \]

\[ \rho z(x^*) \cdot (x - x^*) \geq 0, \]

\[ z(x^*) \cdot (x - x^*) \geq 0, \]  

(21)

To complete this formulation, symmetric matrix \( M \) and positive number \( \rho \) have to be specified. As suggested by Dafermos (1980), \( M \) can be any diagonal matrix (such that the UE flow pattern can be found by the existing algorithm for minimization problem) or as close as possible to Jacobian matrix of link costs (i.e., \( \frac{\partial z}{\partial x} \)). \( \rho \) can be given as \( \mu / \nu \) where \( \mu \) is the minimum eigenvalue of the symmetric part of the Jacobian matrix \( \frac{\partial z}{\partial x} \) of all feasible \( x \), and \( \nu \) is the maximum eigenvalue of the positive definite symmetric matrix \( \left[ \frac{\partial z}{\partial x} \right]^T M^{-1} \left[ \frac{\partial z}{\partial x} \right] \) of all feasible \( x \in X \). The \( M, \mu, \) and \( \nu \) used in the UE analysis are specified in the following Section. Moreover, Dafermos (1980) also provided the proof that with this UE problem based on variational inequalities model, there is an existence of unique solution of UE link flow pattern \( x^* \) if \( M \) is positive definite. In other words, for symmetric matrix \( M, M \) is positive definite if all eigenvalues of \( M \) are positive.

4 Characteristics of User Equilibrium

The formulated variational inequalities-based model on Section 3.2 enables us to mathematically analyzed the existence and uniqueness of UE flow pattern. With the specifications of network condition and ridesharing-related costs in Section 4.1 and specifications of parameters in variational inequalities formulation in Section 4.2, the existence and uniqueness of UE can be mathematically analyzed in Section 4.3.
4.1. Specifications of network condition and ridesharing-related costs

The function of travel time of link $a$ is given as a congestion function developed by Bureau of Public Roads (BPR),

$$ T_a(x_a) = t_a(1 + \alpha_a \left(\frac{x_a}{c_a}\right)^{\beta_a}), $$

(22)

where $t_a$ is a free flow travel time of link $a$, $c_a$ is the capacity of link $a$, and $\alpha_a$ and $\beta_a$ are calibration parameters corresponding to road categories.

Travel fare is a linear increasing function of travel time,

$$ f(T_a(x_a)) = a_f T_a(x_a) + b_f, $$

(23)

where $a_f$ represents an increment rate of fare per one unit of travel time, and $b_f$ represents an initial fare of using a vehicle.

The discomfort cost is also a linear increasing function of travel time,

$$ g(T_a(x_a)) = a_g T_a(x_a) + b_g $$

(24)

where $a_g$ represents the increasing of discomfort cost per one unit of travel time, and $b_g$ represents initial discomfort cost that may occur. The discount rate $r$ is a value between zero and one; and value of time $\sigma$ is positive value.

4.2. Specification of parameters in variational inequalities formulation

The matrix $M$ is selected to be diagonal matrix which is as close to Jacobian matrix of link costs ($\frac{\partial z}{\partial x}$) as possible. Based on the specified functions in Section 4.1, the Jacobian matrix $\frac{\partial z}{\partial x}$ is

$$
\begin{bmatrix}
\frac{\beta_a c_1 x_a^{\beta_a - 1}}{c_1^2} (\sigma + a_f - ra_f + a_g) & 0 & 0 \\
0 & \frac{\beta_a c_1 x_a^{\beta_a - 1}}{c_1^2} (\sigma + a_f) & \frac{\beta_a c_1 x_a^{\beta_a - 1}}{c_1^2} (a_g - ra_f) \\
0 & 0 & 0 \\
\end{bmatrix}
$$

(25)

Then, $M$ is selected as

$$
M =
\begin{bmatrix}
\frac{\beta_a c_1 x_a^{\beta_a - 1}}{c_1^2} (\sigma + a_f - ra_f + a_g) & 0 & 0 \\
0 & \frac{\beta_a c_1 x_a^{\beta_a - 1}}{c_1^2} (\sigma + a_f) & 0 \\
0 & 0 & y \\
\end{bmatrix}
$$

(26)

where $y$ is any small positive number.
To obtain \( \mu \), we have to find the minimum eigenvalue of the symmetric part of the Jacobian matrix of link costs for all feasible link flow patterns. The symmetric part of the Jacobian matrix is
\[
\begin{bmatrix}
\frac{\beta_1 x_t x_{t1}^{\beta_{t1}-1}}{c_1^{\beta_{t1}}} (\sigma + a_f - ra_f + a_g) & 0 & 0 \\
0 & \frac{\beta_2 x_{t2} x_{t2}^{\beta_{t2}-1}}{c_2^{\beta_{t2}}} (\sigma + a_f) & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
and its eigenvalues are \( \frac{\beta_1 x_t x_{t1}^{\beta_{t1}-1}}{c_1^{\beta_{t1}}} (\sigma + a_f - ra_f + a_g) \) and \( \frac{\beta_2 x_{t2} x_{t2}^{\beta_{t2}-1}}{c_2^{\beta_{t2}}} (\sigma + a_f) \).

Therefore,
\[
\mu = \min \left\{ \frac{\beta_1 x_t x_{t1}^{\beta_{t1}-1}}{c_1^{\beta_{t1}}} (\sigma + a_f - ra_f + a_g), \frac{\beta_2 x_{t2} x_{t2}^{\beta_{t2}-1}}{c_2^{\beta_{t2}}} (\sigma + a_f) \right\}
\]
(28)

\( v \) is the maximum eigenvalue of \( \left[ \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right]^T M^{-1} \left[ \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right] \), which must be a positive definite symmetric matrix, of all feasible link flow patterns. From \( M \) in Equation (26) and \( \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \) in Equation (25), \( \left[ \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right]^T M^{-1} \left[ \frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right] \) is
\[
\begin{bmatrix}
\frac{\beta_1 x_t x_{t1}^{\beta_{t1}-1}}{c_1^{\beta_{t1}}} (\sigma + a_f - ra_f + a_g) & 0 & 0 \\
0 & \frac{\beta_2 x_{t2} x_{t2}^{\beta_{t2}-1}}{c_2^{\beta_{t2}}} (\sigma + a_f) & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
(29)

To find eigenvalues, the matrix in Expression (29) is simplified by giving \( A = \frac{\beta_1 x_t x_{t1}^{\beta_{t1}-1}}{c_1^{\beta_{t1}}}, B = \frac{\beta_2 x_{t2} x_{t2}^{\beta_{t2}-1}}{c_2^{\beta_{t2}}}, C = \sigma + a_f, D = -ra_f + a_g \). It can be rewritten as
\[
\begin{bmatrix}
A(C + D) & 0 & 0 \\
0 & B(C) & B(D) \\
0 & B(D) & B(D)^2/(C)
\end{bmatrix}
\]
(30)

The eigenvalues of this matrix are \( (AC + AD + BC + \frac{BD^2}{C}), (BC + \frac{BD^2}{C}) \). To have this matrix as a positive definite, all eigenvalues must be positive. In other words, the following conditions must be satisfied,
\[
BC + \frac{BD^2}{C} > 0,
\]
(31)
\[
AC + AD > -\left( BC + \frac{BD^2}{C} \right).
\]
(32)

Therefore,
\[
v = \max \left\{ \left( AC + AD + BC + \frac{BD^2}{C} \right), (BC + \frac{BD^2}{C}) \right\}
\]
(33)
Since $\mu$ and $\nu$ are already specified, $\rho$ can then be obtained.

4.3. Existence and uniqueness of user equilibrium

As previously mentioned, there is an existence of unique UE solution of link flow pattern if matrix $M$ in Expression (26) is positive definite. Since matrix $M$ is symmetric, $M$ is positive definite if all eigenvalues are positive. In other words, the UE solution is existing and unique if the following conditions are satisfied,

$$\frac{\beta_1 a_1 x_1^{\beta_1 - 1}}{c_1^{\beta_1}} (\sigma + a_f - ra_f + a_g) > 0,$$

(34)

$$\frac{\beta_2 a_2 x_2^{\beta_2 - 1}}{c_2^{\beta_2}} (\sigma + a_f) > 0,$$

(35)

$$\gamma > 0.$$  

(36)

Free flow travel time $t_a$ and capacity $c_a$ of lanes are always positive (in Expressions (34) and (35)). To have a monotonically increasing function of travel time, the calibration parameters $\alpha_a$ and $\beta_a$ must be positive. Therefore, to have a unique UE solution, the following variables and terms must be positive: flow of HOV lane $x_1$, flow of regular lane $x_2$, $\left(\sigma + a_f\right)$, and $\left(\sigma + a_f - ra_f + a_g\right)$. Since the minimum increasing rate of travel fare $a_f$ is zero (i.e., flat rate) and value of time $\sigma$ is always positive, $\left(\sigma + a_f\right)$ is always positive. Moreover, since the minimum possible value of discounted travel fare (i.e., $a_f - ra_f$) is zero (i.e., $r = 1$), and minimum increasing rate of discomfort $a_g$ is zero (i.e., discomfort does not increase over time), the term $\left(\sigma + a_f - ra_f + a_g\right)$ is always positive as well. However, if travelers gain some benefits from ridesharing over time (i.e., negative $a_g$) instead of discomfort, the changing rate of such benefits must satisfy $a_g > -\left(\sigma + a_f - ra_f\right)$ in order to have a unique UE solution. Lastly, value $\gamma$ of matrix $M$ in Expression (26) must always be positive.

5 Discussion and Conclusion

In this study, ridesharing on shared autonomous vehicles with availability of HOV lane was formulated in a traffic equilibrium problem and a variational inequalities problem. The formulation in variational inequalities form has advantages in terms of the ease of mathematical analysis of UE characteristics and UE solution algorithm, especially on large network. The analysis of UE characteristics (e.g., uniqueness of UE solution) is useful for obtaining the insightful implications for ridesharing-related authorities or ridesharing operators. For instance, if the UE solution is not unique, there might be an existence of critical mass of ridesharing travelers to trigger the system to the desirable UE where there are more travelers ridesharing than riding alone. In order to reach the critical mass, some policies might be required.
The ridesharing UE model in this framework can be extended and applied to a general network (e.g., multiple origins and destinations). The extended model and its UE analysis shall be useful for designing the road infrastructure (e.g., the capacity and location of HOV lane that can be efficiently and effectively used by shared autonomous vehicles), deciding the fleet size (e.g., number of vehicles, vehicle capacity).

References


