Centralized and Decentralized Optimal Frequency Setting for Routes Sharing a Transfer Stop

Matan Shnaiderman · Yuval Hadas

Abstract Common practice in public transportation (PT) planning is to determine the frequency of service based on demand, travel time, and given vehicle capacity. With the increased usage of automatic vehicle location (AVL) and automatic passenger counting (APC) systems, it is possible to construct the statistical distributions of passenger demand and travel time by time of day. This can give rise to improve the accuracy of the frequencies determined. Previous study introduced the concept of a supply-chain optimization model for setting optimal frequencies, based on two main cost elements: (a) empty-seat driven (unproductive cost), and (b) overload and unserved demand (increased user cost). The current work extends the model as follows: 1) waiting time costs are included, 2) transfers are considered, and 3) frequency is set jointly for two routes sharing a transfer stop. A numerical example is presented to illustrate the advantages of jointly setting frequencies for two routes as opposed to setting frequencies independently.

Keywords: Public transportation · Frequency setting · Transfers · Optimization

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1 Introduction

Automatic vehicle location (AVL) and automatic passenger counting (APC) are two technologies enabling tracking the location of vehicles in route (AVL), and collecting the number of passengers alighted and boarded at each stop (APC). The data acquired can be used for analysis, as well as to enhance the performance of public transportation (PT) systems, and the introduction of advanced models. Based on AVL technology it is possible to forecast accurately the buses estimated arrival times and to use bus holding strategies to coordinate transfers (Dessouky et al. 2003). The use of advanced PT systems in fixed-route and paratransit operations was found important for improvements in departure times and transfers (Levine et al. 2000). Travel time estimation is also possible (Tétreault and El-Geneidy 2010), as well as the evaluation of PT operations based on both AVL and APC data (Strathman et al. 2002). Advanced supply chain models, specifically inventory models (Lee et al. 2000; Zipkin 2000), are used to optimize the total costs occur within a time frame, by setting inventory replenishment strategies (Kogan and Shnaiderman 2010). The costs are associated with shortage and overage (or surplus). The former is relevant when demand is not met, while the latter when inventory is higher than the demand. Such models can be used to formulate an optimal frequency setting model, in which shortage and overage costs are transformed to overload and empty-seats costs respectively, and inventory strategy transforms to bus capacity (Hadas and Shnaiderman 2012). Furthermore, passengers’ waiting time must be taken into consideration (Peng et al. 2014) as it is part of the total travel time cost (Ceder 2007).

This paper presents a new concept for optimal frequency setting, relies on Hadas and Shnaiderman (2012) supply-chain based model that integrates costs, stochastic demand, and travel time for optimal frequency setting of a single route. The proposed model has the following contributions: 1) waiting time costs are included, and 2) Two routes sharing a transfer stop are considered. Furthermore, two variations of the model are introduced: a) centralized model, in which the frequency is set jointly for the two routes, and b) decentralized model, in which the frequency is set independently for each route.

1 Problem definition

2.1 Assumptions

We consider two routes, illustrated in Fig. 1, each serving passengers at a fixed frequency (or headway) during total time $T$. Vehicles have the same capacity for each route. Let $i$ be the vehicle index, and $j$ the route index ($R_j$ vehicles in route $j$, $j=1,2$). Let $k$ be the stop index ($K_j$ stops in route $j$). $CP_j$ denotes the capacity of the vehicles in route $j$. There is exactly one joint stop in which the two routes intersect, and passengers transfer, say stop $k$, for route $j$. The passenger demand (the unconstrained load) of vehicle $i$ from stop $k$ in route $j$ is denoted by $d_{ik}^j$. We assume that the demand characteristics do not change during the whole period, which was predefined based on demand analysis.
Let $c^+$ be the empty seat overage cost per time unit, let $c^-$ be the un-served passenger shortage cost per time unit, and let $c^w$ denote the unit waiting time cost of one passenger. Furthermore, let $R_{ik}^{(j)}$ be the running time between stop $k$ to stop $k+1$ for vehicle $i$ of route $j$.

The decision variables are: 1) the frequency of each route, which for simplicity, are realized as headways, denoted by $H^{(j)}$ as the headways of route $j$, and 2) the departures’ offset of route $j$, denoted as $o^{(j)}$ (for a centralized system, see below). The offset is the fixed shift in departures from start of service. The offset is aimed at synchronizing the arrival at the transfer stop in order to decrease the transfer time. As the offset is relative, it is required only for one route, hence $o^{(1)}$ is arbitrarily set as the decision variable, while $o^{(2)}=0$. Consequently, there are $\tau^{(j)} = \frac{H^{(j)} - o^{(j)}}{\sum_{j=1}^{J} H^{(j)}}$.

If the demand is smaller than the capacity, then the overage cost of vehicle $i$ of route $j$ at stop $k$ is equal to $R_{ik}^{(j)} c^+ \left(CP^{(j)} - d_{ik}^{(j)}\right)$. On the other hand, if the demand is higher than the capacity, then the shortage cost of vehicle $i$ of route $j$ at stop $k$ is $R_{ik}^{(j)} c^- \left(d_{ik}^{(j)} - CP^{(j)}\right)$. Since we deal with the ratio between the two unit costs, we can assume that $c^+ c^-=1$. The average surplus / shortage cost (per one traveling time unit) is

$$C_1^{(j)} = \frac{\sum_{i=1}^{I^{(j)}} \sum_{k=1}^{K^{(j)}-1} R_{ik}^{(j)} \left[c^+ \left(CP^{(j)} - d_{ik}^{(j)}\right) + c^- \left(d_{ik}^{(j)} - CP^{(j)}\right)\right]}{\sum_{i=1}^{I^{(j)}} \sum_{k=1}^{K^{(j)}-1} R_{ik}^{(j)}}$$

Let $N_{ik}^{(j)}$ be the number of passenger who actually board vehicle $i$ of route $j$ at stop $k$. We denote by $AW_0$ the average waiting time of those passengers. The average waiting time (per passenger) is

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**Fig. 1** Two routes with joint transfer stop
\[
C_2^{(j)} = \frac{\sum_{l=1}^{r_j} \sum_{k=1}^{s_j} c_{wk}^j A_{ik}^j W^{(j)}_{ik}}{\sum_{l=1}^{r_j} \sum_{k=1}^{s_j} N_{ik}^{(j)}}
\]
(\text{where } c_w \text{ is fixed as 1}). Similar to supply chain management models, we consider both "decentralized" and "centralized" systems. In the former, the two routes are operated by two separate companies, and the goal of each is to minimize its own expected cost. Let \(0 \leq \zeta \leq 1\) be the domination of the surplus/shortage cost, then the objective function is
\[
\text{ETC}^{(j)}(H^{(1)}, H^{(2)}) = \sum_{i=1}^{r_j} \sum_{k=1}^{s_j} E \left[ \zeta C_1^{(j)} + (1 - \zeta) C_2^{(j)} | H^{(1)}, H^{(2)} \right]
\]
with decision variables \(\{H^{(1)}, H^{(2)}\}\). The optimal solution then satisfies Nash equilibrium.

On the other hand, in the centralized system, one company operates the two routes, or the PT authority enforces routes' synchronization, in order to minimize the sum of the expected costs. The objective function is the expected total cost for all vehicles and stops:
\[
\text{ETC}(o^{(1)}, H^{(1)}, H^{(2)}) = \sum_{j=1}^{2} \sum_{i=1}^{r_j} \sum_{k=1}^{s_j} E \left[ \zeta C_1^{(j)} + (1 - \zeta) C_2^{(j)} | o^{(1)}, H^{(1)}, H^{(2)} \right]
\]
with decision variables \(\{o^{(1)}, H^{(1)}, H^{(2)}\}\).

Let \(L_{ik}^{(j)}\) be the constrained load in vehicle \(i\) of route \(j\) after departing from stop \(k\), that is, \(L_{ik}^{(j)} = \min(d_{ik}^{(j)}, C^j(P))\). Let \(A_{ik}^{(j)}\) denote the number of passengers alighting from vehicle \(i\) of route \(j\) at stop \(k\) \((A_{i1}^{(j)}\) is assumed to be zero), and \(B_{ik}^{(j)}\) the number of passengers of route \(j\) arriving at stop \(k\) since vehicle \(i-1\) has left until vehicle \(i\) arrives. Also, let \(U_{ik}^{(j)}\) be the number of un-served passengers by vehicle \(i\) of route \(j\) at stop \(k\), that is, \(U_{ik}^{(j)} = \left(d_{ik}^{(j)} - C^j(P)\right)^+\).

### 2.2 One-Route Stops

#### 2.2.1 Calculation of the demands

Setting \(d_{ik}^{(j)} = 0\) as well as \(d_{0k}^{(j)} = 0\), the following formula is derived for the demand \(d_{ik}^{(j)}\) while \(k \neq k_j\),
\[
d_{ik}^{(j)} = L_{ik-1}^{(j)} - A_{ik}^{(j)} + u_{i-1,k}^{(j)} + B_{ik}^{(j)}.
\]
Let \(D_{ik}^{(j)}\) be the dwell time of vehicle \(i\) of route \(j\) at stop \(k\), and let \(H_{ik}^{(j)}\) be the headway between the departure times of vehicles \(i-1\) and \(i\) of route \(j\) at stop \(k\), then
\[
\begin{align*}
H_{i1}^{(j)} &= H^{(j)} + D_{i1}^{(j)} - D_{i-1,1}^{(j)}, \\
H_{ik}^{(j)} &= H_{i,k-1}^{(j)} + R_{i,k-1}^{(j)} - R_{i-1,k-1}^{(j)} + D_{ik}^{(j)} - D_{i-1,k}^{(j)}.
\end{align*}
\]

Moreover, let \(a, b_a, \) and \(b_b\) be lost time due to accelerating and decelerating of the vehicles at each stop, the time for a single passenger to alight from the vehicle and the time for a single passenger to board the vehicle respectively. The dwell time \(D_{ik}^{(j)}\) (while \(k\neq k_j\)) is equal to

\[
D_{ik}^{(j)} = a + b_a A_{ik}^{(j)} + b_b N_{ik}^{(j)}
\]

(Sun and Hickman 2008). Denote by \(Q_{ik}^{(j)}\) the available number of passengers that may board vehicle \(i\) of route \(j\) at stop \(k\), that is, \(Q_{ik}^{(j)} = \text{CP}_{\{L_{j,k}, i\}}(A_{k}^{(j)})\), then \(N_{ik}^{(j)} = \min \left( U_{i-1,k}^{(j)} + B_{ik}^{(j)} , Q_{ik}^{(j)} \right)\)

### 2.2.2 Calculation of the average waiting time

Next, we consider the passengers' average waiting time. Let \(1 \leq j \leq 2\), \(1 \leq i \leq I\) and \(1 \leq k \leq K-1\) such that \(k \neq k_j\). As aforesaid, the number of passengers that were unserved by the previous vehicle, \(i-1\), is \(U_{i-1,k}^{(j)}\). This value may be factored to the following sum \(U_{i-1,k}^{(j)} = \sum_{m=1}^{i-1} U_{i-1,k,m}^{(j)}\), where \(U_{i-1,k,m}^{(j)}\) denotes the number of passengers that arrived to stop \(k\) between the departures of vehicles \(m-1\) and \(m\), and were unserved by vehicles \{\(m,\ldots,i-1\}\). Also, let \(SU_{i-1,k,m}^{(j)}\) be the set of those passengers, i.e.

\[
\left| SU_{i-1,k,m}^{(j)} \right| = U_{i-1,k,m}^{(j)}.
\]

In other words, \(SU_{i-1,k,m}^{(j)}\) is a subset of \(SB_{mk}^{(j)}\), where the latter denotes the set of all \(B_{mk}^{(j)}\) passengers that arrived to stop \(k\) between the departures of vehicles \(m-1\) and \(m\). Moreover, its \(U_{i-1,k,m}^{(j)}\) passengers chronologically arrived to the bus stop after the other \(B_{km}^{(j)} - U_{i-1,k,m}^{(j)}\) passengers of \(SB_{mk}^{(j)}\), who have been served by earlier vehicles than \(i\), did. We also denote \(SU_{i-1,k}^{(j)} = \bigcup_{m=1}^{i-1} SU_{i-1,k,m}^{(j)}\) (namely, \(\left| SU_{i-1,k}^{(j)} \right| = U_{i-1,k}^{(j)}\)). We call here the passengers from \(SU_{i-1,k}^{(j)}\) and \(SB_{ik}^{(j)}\) "old" and "new" passengers, respectively, on behalf of vehicle \(i\) at stop \(k\).

One of the two following situations is now valid:

- We have \(U_{i-1,k}^{(j)} \leq Q_{ik}^{(j)}\) .

This situation means that all unserved passengers of the previous vehicles at stop \(k\) (old passengers) may now board vehicle \(i\), and there is available space for additional \(Q_{ik}^{(j)} - U_{i-1,k}^{(j)}\) new passengers from \(SB_{ik}^{(j)}\) (who arrived to that stop after vehicle \(i-1\) had left it). The actual number of passengers that board the vehicle at this stop is \(N_{ik}^{(j)} = \min \left( U_{i-1,k}^{(j)} + B_{ik}^{(j)} , Q_{ik}^{(j)} \right)\).
- There exists a unique $0 \leq i \leq i-2$ such that
\[ 0 \leq \sum_{m=1}^{i_1} U_{i-1,km}^{(j)} \leq Q_{ik} < \sum_{m=1}^{i_1+1} U_{i-1,km}^{(j)}. \] (5)
That is, all passengers of the union $\bigcup_{m=1}^{i_1} SU_{i-1,km}^{(j)}$ are boarding vehicle $i$. In addition, only $Q_{ik} - \sum_{m=1}^{i_1} U_{i-1,km}^{(j)}$ passengers from $SU_{i-1,k,i+1}^{(j)}$ (namely, less than $U_{i-1,k,i+1}^{(j)}$) are boarding. The rest passengers of $SU_{i-1,k,i+1}^{(j)}$ as well as all those of $\bigcup_{m=i_1+2}^{i_1+k} U_{i-1,km}^{(j)}$ remain unserved by vehicle $i$. Moreover, all new passengers at stop $k$ ($B_{ik}$ passengers) become unserved by that vehicle. In particular, if $i=0$, then the available capacity of vehicle $i$ at stop $k$ is not enough even for all the “oldest” unserved passengers, those of $U_{i-1,k,1}$.

The actual number of passengers that board the vehicle is $N_{ik}^{(j)} = Q_{ik}^{(j)}$.

If $N_{ik}^{(j)} = 0$, we set $AW_{ik}^{(j)} = 0$. Therefore, we now consider the case of $N_{ik}^{(j)} > 0$. Assuming that passengers arrive to each stop in fixed rate, the average waiting time of passengers that actually board vehicle $i$ at stop $k \neq k_0$ is calculated according to proposition 1 below. Taking into account cases of zero people who arrive to the stations during time intervals, we define binary variables as follows. For every $1 \leq m \leq i$, define $\delta_{mk}^{(j)} = 1$ if $B_{mk}^{(j)} = 0$ and $\delta_{mk}^{(j)} = 0$ otherwise, (note that if $B_{mk}^{(j)} = 0$ then $U_{i-1,km}^{(j)} = 0$, and that (5) leads to $B_{i+1,k} > 0$).

**Proposition 1:** If (4) is satisfied then
\[
AW_{ik}^{(j)} = \frac{1}{\min \left\{ U_{i-1,k}^{(j)} + B_{ik}, Q_{ik} \right\}} \times \left\lfloor \sum_{m=1}^{i_1} U_{i-1,km}^{(j)} \left( \frac{U_{i-1,km}^{(j)}}{2B_{mk}^{(j)} + \delta_{mk}^{(j)}} H_{ik}^{(j)} + \sum_{m=1}^{i} H_{ik}^{(j)} \right) + \min \left\{ B_{ik}^{(j)}, Q_{ik}^{(j)} - U_{i-1,k}^{(j)} \right\} \times \left( 1 - \frac{\min \left\{ B_{ik}^{(j)}, Q_{ik}^{(j)} - U_{i-1,k}^{(j)} \right\}}{2B_{ik}^{(j)} + \delta_{ik}^{(j)}} \right) H_{ik}^{(j)} \right\rfloor. \] (6)

Under (5),
\[
AW_{ik}^{(j)} = \frac{1}{Q_{ik}} \times \left\lfloor \sum_{m=1}^{i_1} U_{i-1,km}^{(j)} \left( \frac{U_{i-1,km}^{(j)}}{2B_{mk}^{(j)} + \delta_{mk}^{(j)}} H_{ik}^{(j)} + \sum_{m=1}^{i} H_{ik}^{(j)} \right) + \left( Q_{ik}^{(j)} - \sum_{m=1}^{i_1} U_{i-1,km}^{(j)} \right) \times \left( \frac{2U_{i-1,k,i+1}^{(j)} - \left( Q_{ik}^{(j)} - \sum_{m=1}^{i_1} U_{i-1,km}^{(j)} \right) H_{i+1,k}^{(j)} + \sum_{m=1}^{i} H_{ik}^{(j)}}{2B_{i+1,k}} \right) \right\rfloor. \] (7)
\[\square\]
**Proof:** See in the Appendix.

**Remark 1:** According to our assumption, passengers board the vehicles following their arrival order (FCFS). Therefore, if \( U_{i-1,km}^{(j)} > 0 \) then \( U_{i-1,kf}^{(j)} = B_{tk}^{(j)} \) for every \( m+1 \leq f < j-1 \).

### 2.3 Transfer Stop

Let \( j \in \{1, 2\} \), and denote by \((-j)\) the other route. Let \( T_{i}^{(j)} \) be the number of passengers that alight from vehicle \( i \) of route \( j \) at stop \( k_j \) and are interested to board a vehicle of route \((-j)\) there, and let \( ST_{i}^{(j)} \) be the set of those passengers. For every \( 1 \leq i \leq P \) and \( 1 \leq j \leq P \), denote by \( h_{i_1i_2}^{(j)} \) the headway between the arrival time of vehicle \( i_2 \) of route \((-j)\) at stop \( k_j \) and the departure time of vehicle \( i_1 \) of route \( j \) at that stop, setting \( h_{i_10}^{(j)} = H_{i_1k_j}^{(j)} \). For \( 1 \leq j \leq P \) we define the following set

\[
S_{l}^{(j)} = \left\{ 1 \leq l \leq I^{(-j)} \mid 0 \leq h_{il} \leq H_{ik_j} \right\},
\]

which is the set of all vehicles of route \((-j)\) that arrive to the transfer stop between the departures of vehicles \( i-1 \) and \( i \) of route \( j \) there.

If \( S_{l}^{(j)} \neq \emptyset \), we define \( i_{l}^{(-j)}(i) = \min S_{l}^{(j)} \) as well as \( i_{l}^{(-j)}(i) = \max S_{l}^{(j)} \) (here \( f \) and \( l \) refer to "first" and "last" respectively), that is,

\[
h_{u_{l}^{(-j)}(i)}^{(j)} \leq h_{u_{l}^{(-j)}(i)}^{(j)} \leq \cdots \leq h_{u_{l}^{(-j)}(i)}^{(j)} \leq H_{ik_j} < h_{u_{l}^{(-j)}(i)}^{(j)} \leq h_{u_{l}^{(-j)}(i)}^{(j)},
\]

and \( S_{l}^{(j)} = \{ i_{l}^{(-j)}(i), \ldots, i_{l}^{(-j)}(i) \} \). Condition (8) means that during headway \( H_{ik_j}^{(j)} \), i.e. between the departures of stop \( k_j \) of vehicles \( i \) and \( i-1 \) of route \( j \), passengers of vehicles of route \((-j)\) from \( S_{l}^{(j)} \) (i.e. \( i_{l}^{(-j)}(i) - i_{l}^{(-j)}(i) + 1 \) vehicles) alight at stop \( k_j \) (or \( k_{j-1} \), on behalf of route \((-j)\)), and some of them are interested to board vehicle \( i \) of route \( j \) at that stop. On the other hand, vehicle \( i_{l}^{(-j)}(i) - 1 \) of route \((-j)\) that had arrived to stop \( k_j \) before vehicle \( i-1 \) of route \( j \) departed that stop (that is, the passengers of \( ST_{l}^{(j)} \) \( i_{l}^{(-j)}(i) - 1 \) try first to board earlier vehicles than \( i \) of route \( j \)). In addition, vehicle \( i_{l}^{(-j)}(i) + 1 \) of route \((-j)\) had not arrived to stop \( k_j \) before vehicle \( i \) of route \( j \) departed that stop (and the passengers of \( ST_{l}^{(j)} \) \( i_{l}^{(-j)}(i) + 1 \) try to board latter vehicles than \( i \) of route \( j \)).

If \( S_{l}^{(j)} = \emptyset \), then we define \( i_{l}^{(-j)}(i) = i_{l}^{(-j)}(i) = I^{(-j)} + 1 \). Also, we set \( h_{u_{l}^{(-j)}(i)+1}^{(j)} = H_{ik_j} \) as well as \( h_{u_{l}^{(-j)}(i)+2}^{(j)} = 0 \).

### 2.3.1 Demands

Fix \( T_{l}^{(-j)} = 0 \), then the demand for vehicle \( i \) of route \( j \) at stop \( k_j \) is,
\[ d_{ikj}^{(j)} = t_{ikj-1}^{(j)} - A_{ikj}^{(j)} + U_{-1,ikj}^{(j)} + B_{ikj}^{(j)} + \sum_{l=\max(l-1,j)}^{l_1^{(j)}} T_l^{(-j)}. \]

Given \( l_1^{(-j)}(i) \) and \( l_1^{(-j)}(i) \), for every \( 0 \leq i \leq f^0 \), we can write \( B_{ikj}^{(j)} = \sum_{n=\max(n,f^0)}^{l_1^{(-j)}(i)} B_{ikj,n}^{(j)} + B_{ikj,l_1^{(-j)}(i)+1}^{(j)}, \) such that for every \( l_1^{(-j)}(i) < n \leq l_1^{(-j)}(i) \), \( B_{ikj,n}^{(j)} \) is the number of passengers that independently arrive to stop \( k \), in order to board vehicle \( i \) of route \( j \). 

The dwell time of that vehicle at stop at that stop is \( U_{ikj}^{(j)} \), and we now consider the case of \( N_{ikj}^{(j)} \) respectively, who were unserved by vehicles \( \{m, \ldots, i\} \) of route \( j \).

The dwell time of that vehicle at stop at that stop is

\[ D_{ikj}^{(j)} = a + bA_{ikj}^{(j)} + bB_{ikj}^{(j)}. \]

### 2.3.2 Average Waiting Time

The average waiting time at stop \( k_j \) is calculated as follows. As before, if \( N_{ikj}^{(j)} = 0 \), we set \( AW_{ikj}^{(j)} = 0 \), and we now consider the case of \( N_{ikj}^{(j)} > 0 \). As before, we define
Next, we assume that possible. In the first one, there exists \( i \) such that the available number of passengers to board vehicle \( i \) at stop \( k \), \( Q_n \), is higher than the total number of old unserved passengers by vehicle \( i-1 \), and all new passengers from \( SB_{ik_j} \). Hence, \( N_{ik_j}^{(i)} = U_{i-1,k_j}^{(i)} + B_{ik_j}^{(i)} + \sum_{m=1}^{i-1} t_m^{(-i)} T_n^{(-i)} \).

**Proposition 2:** If (9) is satisfied, then the average waiting time of passengers who actually board vehicle \( i \) of route \( j \) at stop \( k \) is

\[
AW_{ik_j}^{(i)} = \frac{1}{U_{i-1,k_j}^{(i)} + B_{ik_j}^{(i)} + \sum_{m=1}^{i-1} t_m^{(-i)} T_n^{(-i)}} \left( U_{ik_j}^{(i)} \frac{U_{ik_j,1,k_j}^{(i)}}{2B_{ik_j,n}^{(i)} + h_{ik_j}^{(i)}} \right) + \left( h_{min} + \sum_{\ell=m+1}^{i} H_{ik_j}^{(i)} \right) + \left( U_{i-1,k_j,1}^{(i)} \frac{U_{i-1,k_j,1,k_j}^{(i)} + h_{ik_j}^{(i)}}{2B_{ik_j,n}^{(i)} + h_{ik_j}^{(i)}} \right) + \left( h_{min} \frac{H_{ik_j}^{(i)}}{2} + T_n^{(-i)} \right) + \frac{h_{ik_j}^{(i)} h_{ik_j}^{(i)}}{2}. \tag{10}
\]

Next, we assume that \( 0 \leq Q_{ik_j}^{(i)} - U_{i-1,k_j}^{(i)} < B_{ik_j}^{(i)} + \sum_{m=1}^{i-1} t_m^{(-i)} T_n^{(-i)} \). This situation means that vehicle \( i \) has enough space to serve all old passengers of \( SU_{i-1,k_j}^{(i)} \), but only some of the new passengers in \( SB_{ik_j}^{(i)} \). Two positions are now possible. In the first one, there exists \( i \) such that

\[
0 \leq Q_{ik_j}^{(i)} - U_{i-1,k_j}^{(i)} + \sum_{n=1}^{i-1} \left( B_{ik_j,n}^{(i)} + T_n^{(-i)} \right) < B_{ik_j}^{(i)} + 1. \tag{11}
\]
In the other position, there exists $i'_{T}(i) - 1 \leq i_0 \leq i'_{T}(i) - 1$ such that

$$0 \leq Q_{i_0}^{(i)} - \left[ u_{i-1,h_{i}}^{(i)} + \sum_{n=\lceil i'_{T}(i) \rceil}^{i_0} \left( B_{i,n}^{(i)} + T_{i}^{(i)} \right) + B_{i_0,i+1}^{(i)} \right] < T_{i+1}^{(i)}. \quad (12)$$

**Proposition 3:** If (11) is met, then

$$AW_{i_0}^{(i)} = \begin{cases} 
\sum_{j=1}^{l} \left[ \sum_{n=j}^{i_0} \left( u_{i-1,j}^{(i)} + \min\left( h_{i,n-1}^{(i)}, h_{i,n}^{(i)} \right) \right) \left( B_{i,n}^{(i)} + T_{i}^{(i)} \right) \right] + h_{i_0}^{(i)} + \sum_{\ell=m+1}^{i_0} H_{\ell}^{(i)} \\
+ U^{(i)}_{T-1,i_0,m} \left( h_{i_0}^{(i)} + \sum_{\ell=m+1}^{i} H_{\ell}^{(i)} \right) \\
+ \sum_{n=\lceil i'_{T}(i) \rceil}^{i_0} \left[ B_{i,n}^{(i)} + h_{i,n}^{(i)} \right] + \sum_{\ell=m+1}^{i} H_{\ell}^{(i)} \\
+ \left[ Q_{i_0}^{(i)} - \left( u_{i-1,j}^{(i)} + \sum_{n=\lceil i'_{T}(i) \rceil}^{i_0} \left( B_{i,n}^{(i)} + T_{i}^{(i)} \right) \right) \right] \times \left[ 1 - \frac{\min\left( h_{i_0}^{(i)}, h_{i_0}^{(i)} \right) - \max\left( h_{i_0}^{(i)}, 0 \right)}{2 B_{i_0,i+1}} \right] \\
+ \max(h_{i_0,i+1}^{(i)}) \end{cases}. \quad (13)$$

Under (12),

$$AW_{i_0}^{(i)} = \begin{cases} 
\sum_{j=1}^{l} \left[ \sum_{n=j}^{i_0} \left( u_{i-1,j}^{(i)} + \min\left( h_{i,n-1}^{(i)}, h_{i,n}^{(i)} \right) \right) \left( B_{i,n}^{(i)} + T_{i}^{(i)} \right) \right] + h_{i_0}^{(i)} + \sum_{\ell=m+1}^{i_0} H_{\ell}^{(i)} \\
+ U^{(i)}_{T-1,i_0,m} \left( h_{i_0}^{(i)} + \sum_{\ell=m+1}^{i} H_{\ell}^{(i)} \right) \\
+ \sum_{n=\lceil i'_{T}(i) \rceil}^{i_0} \left[ B_{i,n}^{(i)} + h_{i,n}^{(i)} \right] + \sum_{\ell=m+1}^{i} H_{\ell}^{(i)} \\
+ \left[ Q_{i_0}^{(i)} - \left( u_{i-1,j}^{(i)} + \sum_{n=\lceil i'_{T}(i) \rceil}^{i_0} \left( B_{i,n}^{(i)} + T_{i}^{(i)} \right) \right) \right] \times \left[ 1 - \frac{\min\left( h_{i_0}^{(i)}, h_{i_0}^{(i)} \right) - \max\left( h_{i_0}^{(i)}, 0 \right)}{2 B_{i_0,i+1}} \right] \\
+ \max(h_{i_0,i+1}^{(i)}) \end{cases}. \quad (14)$$
Note that if no vehicles of route \((j)\) arrive to stop \(k_j\) while vehicle \(i\) of route \(j\) dwells there (that is, \(i_{(j)}^{(i)}(i) = i_{(j)}^{(i)}(i) = I^{(j)} + 1\)), then formulas (13)-(14) coincide with formula (6).

Finally, we consider the case of \(U_{i-1,k_j}^{(j)} > Q_{i,k_j}^{(j)}\), in which there is no available space in vehicle \(i\) of route \(j\) even for all the old passengers in stop \(k_j\). As a result, only some of them board the vehicle, and all the new passengers remain unserved. There exists \(0 \leq i \leq i - 2\) such that \(\sum_{m=1}^{i_1} u_{i-1,k_j,m}^{(j)} \leq Q_{i,k}^{(j)} < \sum_{m=1}^{i_1+1} u_{i-1,k_j,m}^{(j)}\). Namely, all unserved passengers from \(U_{i-1,k_j,i+1}^{(j)}\) board vehicle \(i\). The rest \(U_{i-1,k_j,i+1}^{(j)} - \left(Q_{i,k}^{(j)} - \sum_{m=1}^{i_1} u_{i-1,k_j,m}^{(j)}\right)\) passengers from \(SU_{i-1,k_j,i+1}^{(j)}\) as well as all those of \(U_{i-1,k_j,i}^{(j)}\) remain unserved. Again, two situations are possible now. Either there exists \(i_{(j)}^{(i)}(i_1 + 1) - 1 \leq i_0 \leq i_{(j)}^{(i)}(i_1 + 1)\) such that

\[
0 \leq Q_{i,k}^{(j)} = \left[\sum_{m=1}^{i_1} u_{i-1,k_j,m}^{(j)} + \sum_{n=i_1}^{i_0} \left(u_{n-1,k_j,i+1}^{(j)} + u_{n-1,k_j,i+1}^{(j)}\right)\right] \leq u_{i-1,k_j,i+1}^{(j)}, \tag{15}
\]

or there exists \(i_{(j)}^{(i)}(i_1 + 1) - 1 \leq i_0 \leq i_{(j)}^{(i)}(i_1 + 1) - 1\) such that

\[
0 \leq Q_{i,k}^{(j)} = \left[\sum_{m=1}^{i_1} u_{i-1,k_j,m}^{(j)} + \sum_{n=i_1}^{i_0} \left(u_{n-1,k_j,i+1}^{(j)} + u_{n-1,k_j,i+1}^{(j)}\right)\right] < u_{i-1,k_j,i+1}^{(j)}. \tag{16}
\]

The average waiting time \(AW_{i,k}^{(j)}\) now becomes as follows.

**Proposition 4:** If (15) is satisfied, then
\[
\begin{align*}
\text{AW}^{(j)}_{\text{IA}ij} &= \frac{1}{Q_{\text{IA}ij}} \left( \sum_{n=1}^{i} \sum_{\gamma_i \in \gamma_i} \left[ \sum_{m=i}^{i} \left( U^{(j)}_{\text{IA}i+1,j,m} \left( \frac{U^{(j)}_{\text{IA}i+1,j,m} \min(h^{(j)}_{m,n-1}, H^{(j)}_{m,k_n}) - h^{(j)}_{m,n-1}}{2} + b^{(j)}_{m} + \sum_{k=m+1}^{i} H^{(j)}_{k_i} \right) \\
+ U^{(j)}_{\text{IA}i+1,j,m} \left( \frac{U^{(j)}_{\text{IA}i+1,j,m} \min(h^{(j)}_{m,n-1}, H^{(j)}_{m,k_n}) - h^{(j)}_{m,n-1}}{2} + \sum_{k=m+1}^{i} H^{(j)}_{k_i} \right) \right) \right]
+ \sum_{n=i+1}^{i} \min(h^{(j)}_{i+1,n}, H^{(j)}_{i+1,n})
+ \sum_{n=i+1}^{i} \min(h^{(j)}_{i+1,n}, H^{(j)}_{i+1,n})
\right]\right)
+ U^{(j)}_{\text{IA}i+1,j,m} \left( \frac{U^{(j)}_{\text{IA}i+1,j,m} \min(h^{(j)}_{m,n-1}, H^{(j)}_{m,k_n}) - h^{(j)}_{m,n-1}}{2} + \sum_{k=m+1}^{i} H^{(j)}_{k_i} \right) \right]
+ \sum_{n=i+1}^{i} \min(h^{(j)}_{i+1,n}, H^{(j)}_{i+1,n})
+ \sum_{n=i+1}^{i} \min(h^{(j)}_{i+1,n}, H^{(j)}_{i+1,n})
\right]\right)
+ \sum_{n=i+1}^{i} \min(h^{(j)}_{i+1,n}, H^{(j)}_{i+1,n})
+ \sum_{n=i+1}^{i} \min(h^{(j)}_{i+1,n}, H^{(j)}_{i+1,n})
\right]\right)
\end{align*}
\]

(17)

On the other hand, (16) leads to
As before, if \( i_f^{(-j)}(i) = i_f^{(-j)}(i) + 1 \), then formulas (17)-(18) become (7).

4 Numerical Examples and Sensitivity Analysis

In order to illustrate our results, a numerical example is presented, with the following assumptions:

- The number of passengers arrive to each stop (excluding route-transfer), \( B_k(i,j) \), satisfies a Poisson process.
- Given the load \( L_{i,k}(i,j) \), the number of passengers that alight at each stop, \( A_{i,k}(i,j) \), is binomially distributed with parameter \( p_k(i,j) \). Also, given \( A_{i,k}(i,j) \) the number of passengers that alight vehicle \( i \) of route \( j \) at stop \( k \) in order to board a vehicle of route \( k \), \( T_k(i,j) \), is binomially distributed as well, with parameter \( q_k(i,j) \).
For every fixed $j$ and $k$, the running times $\{R_{ik}(j)\}$ satisfy an autoregressive process AR(1) (Hadas and Shnaiderman 2012; Mishalani et al. 2008). That is,
\[
\begin{align*}
R_{ik}(j) &= \mu_k^{(j)}(j) + \epsilon_{ik}(j) \\
R_{ik}(j) &= (1 - \alpha)\mu_k^{(j)} + \alpha R_{i,k-1}(j) + \epsilon_{ik}(j), \quad 2 \leq i \leq I^{(j)}
\end{align*}
\]
where $0 \leq \alpha \leq 1$ is the correlation level, $\mu_k^{(j)}>0$ is the non-conditional average running time of route $j$ from stop $k$ to stop $k+1$, and the random variable $\epsilon_{ik}(j)$ (the white noise) is normally distributed with mean 0 and standard deviation $\sigma$ (for simplicity, we assume in the current example that $\alpha$ as well as $\sigma$ does not depend on $k$ or $j$).

The minimal and maximal possible headways are 1 minute and 10 minutes respectively.

Furthermore, the parameters were set to (all time units are minutes):
- $K(1)=K(2)=7$, $k_1=k_2=4$;
- $T=60$ ;
- $a=0$, $b_A=1/60$, $b_B=1/30$, $b_T=1/6$ ;
- $CP(1)=CP(2)=50$;
- $\lambda^{(1)}=[2,2,4,1.6,0.7,0.2]$, $\lambda^{(2)}=[4/3,8/3,2,1,0.4,0.7]$;
- $p^{(1)}=[0.01,0.5,0.2,0.15,0.05,0.16]$, $p^{(2)}=[0.01,0.5,0.4,0.3,0.08,0.16]$;
- $q^{(1)}=0.5$, $q^{(2)}=0.3$;
- $\mu^{(1)}=[5,8,7,12,3,5,1.2]$, $\mu^{(2)}=[4,10,5,4.1,12/3,8]$;
- $\alpha=0.5$, $\sigma=0.4$;
- $c^+ = 0.5$, $c^- = 0.5$, $\zeta = 0.5$.

Our sensitivity analysis is presented in Table 1 below. The left three columns refer to the ratio between the unit surplus and shortage costs. While $c^+=0$, only shortage seats and waiting time costs exists, and consequently, the optimal headways are all equal to the minimal value 1. As $c^+$ grows, the optimal headways increase and these of route 1 are always greater than those of route 2. In particular, when $c^+$ becomes greater than or equal to 0.7, the optimal headway of route 2 is equal to the maximal possible value 10. Similar results are obtained while the parameter $\zeta$ runs between 0 and 1, as shown in the middle three columns of Table 1. The right three columns of Table 1 refer to the domination of the transfer stop (4). The probability $q_1$ as well as $q_2$ grows from 0 (i.e. no passengers transfer from one route to the other) to 1 (namely, all passengers who alight from vehicles of one route in the transfer stop transfer to the other route there. The optimal headways generally decrease but are not significantly affected by the that domination. In particular, they always remain higher than or equal to 6 (route 1) and 8 (route 2).

According to Table 1, the optimal headways corresponding to the centralized system are usually the same as those of the decentralized system. In the few exceptional cases, the difference between $H^{(2)*}$ and $H^{(1)*}$ are usually higher in the centralized case. For instance, when $c^+=0.9$, the optimal values are 7 and 10 in the centralized case, and
they become closer, 8 and 9, in the decentralized system. Finally, it is interesting to note that in the decentralized case, the optimal headways of each route do not necessarily increase/decrease as the parameters are changed. Furthermore, whenever the solution is different between the centralized and decentralized systems, the gap to global optimality is less than 9%. This means that utilizing the faster decentralized system, does not increases the costs significantly.

Table 1 Results of the centralized and decentralized systems

<table>
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<tr>
<th>(c,c')</th>
<th>Centralized (H(1)∗,H(2)∗)</th>
<th>Decentralized (H(1)∗,H(2)∗)</th>
<th>ζ</th>
<th>Centralized (H(1)∗,H(2)∗)</th>
<th>Decentralized (H(1)∗,H(2)∗)</th>
<th>q3%</th>
<th>Centralized (H(1)∗,H(2)∗)</th>
<th>Decentralized (H(1)∗,H(2)∗)</th>
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<td>(0.1)</td>
<td>(1.1)</td>
<td>0.1</td>
<td>(6,10)</td>
<td>(7,9) 3%</td>
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<tr>
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<tr>
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<td>(5,8)</td>
<td>(6,7)* 9%</td>
<td></td>
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<tr>
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References

Appendix

**Proof of Proposition 1**: Let $1 \leq m \leq i - 1$, then $SU^{(j)}_{i-1,km}$ contains the passengers of $SB^{(j)}_{mk}$, who arrived during the last $H^{(j)}_{mk}$ minutes (out of $H^{(j)}_{mk}$) of the time between the departures of vehicles $m - 1$ and $m$. If (4) is satisfied, then all those passengers board vehicle $i$ at stop $k$. Consequently, their average waiting time is first (before vehicle $m$ departs) half of that duration. In addition, all those $U^{(j)}_{i-1,km}$ passengers wait at stop $k$ until vehicles $\{m+1,...,i\}$ departed that stop, namely, $\sum_{j=m+1}^{i} H^{(j)}_{ik}$ minutes. Hence, the total waiting time of the passengers of $SU^{(j)}_{i-1,k}$ is $\sum_{j=m+1}^{i} U^{(j)}_{i-1,km} \left( \frac{u^{(j)}_{i-1,km}}{H^{(j)}_{mk}} \right) + H^{(j)}_{mk}$. In addition, $\min \left\{ B^{(j)}_{ik}, Q^{(j)}_{ik} - U^{(j)}_{i-1,k} \right\}$ new passengers board vehicle $i$ at stop $k$. Chronologically, these passengers had arrived to stop $k$ before the rest $B^{(j)}_{ik} - \min \left\{ B^{(j)}_{ik}, Q^{(j)}_{ik} - U^{(j)}_{i-1,k} \right\}$ of $SB^{(j)}_{ik}$ did. Therefore, they first wait, in average, half of their total arrival duration, namely, $\frac{\min \left\{ B^{(j)}_{ik}, Q^{(j)}_{ik} - U^{(j)}_{i-1,k} \right\}}{2} H^{(j)}_{ik}/2$. Then, all of them wait the whole arrival duration of the rest $B^{(j)}_{ik} - \min \left\{ B^{(j)}_{ik}, Q^{(j)}_{ik} - U^{(j)}_{i-1,k} \right\}$ passengers, which is equal to $\frac{B^{(j)}_{ik} - \min \left\{ B^{(j)}_{ik}, Q^{(j)}_{ik} - U^{(j)}_{i-1,k} \right\}}{B^{(j)}_{ik} + \delta^{(j)}_{ik}} H^{(j)}_{ik}$. Consequently, the average waiting time of the passengers who actually board vehicle $i$ at stop $k$ is
\[
AW^{(j)}_{ik} = \frac{1}{\min\{U_{i-1,k}^{(j)} + B_{ik}, Q_{ik}\}} \times \\
\left[ \sum_{m=1}^{i} U_{i-1,k}^{(j)} \left( \frac{u_{i-1,km}^{(j)}}{2B_{mk}^{(j)} + e_{mk}^{(j)}} H_{mk}^{(j)} + \sum_{\ell=m+1}^{i} H_{\ell k}^{(j)} \right) + \min\{B_{ik}^{(j)}, Q_{ik}^{(j)} - U_{i-1,k}^{(j)}\} \times \\
\frac{\min\{b_{ik}^{(j)}, u_{i-1,k}^{(j)}\}}{b_{ik}^{(j)} + e_{ik}^{(j)}} H_{ik}^{(j)} \right],
\]

which can be simplified to (6).

Now assume that (5) is met. For every \(1 \leq m \leq i\), all the \(U_{i-1,k}^{(j)}\) old passengers of STU \(l-1,km\) board vehicle \(i\), and their total arrival waiting time is, as before,

\[
\sum_{m=1}^{i} U_{i-1,k}^{(j)} \left( \frac{u_{i-1,km}^{(j)}}{2B_{mk}^{(j)} + e_{mk}^{(j)}} H_{mk}^{(j)} + \sum_{\ell=m+1}^{i} H_{\ell k}^{(j)} \right) + \min\{B_{ik}^{(j)}, Q_{ik}^{(j)} - U_{i-1,k}^{(j)}\} \times \\
\frac{\min\{b_{ik}^{(j)}, u_{i-1,k}^{(j)}\}}{b_{ik}^{(j)} + e_{ik}^{(j)}} H_{ik}^{(j)}
\]

Then, additional \(Q_{ik}^{(j)} - \sum_{m=1}^{i} U_{i-1,k}^{(j)}\) passengers from STU \(i-1,k,i+1\) (out of \(U_{i-1,k,i+1}^{(j)}\)) board the vehicle. They wait (in average) half of their arrival time duration, namely, \(\frac{Q_{ik}^{(j)} - \sum_{m=1}^{i} U_{i-1,k}^{(j)}}{B_{i+1,k}}\) minutes. Then, all of them wait the total arrival duration of the rest \(U_{i-1,k,i+1}^{(j)} - (Q_{ik}^{(j)} - \sum_{m=1}^{i} U_{i-1,k}^{(j)})\) passengers of STU \(i-1,k,i+1\), which is equal to \(\frac{u_{i-1,k,i+1}^{(j)} - (Q_{ik}^{(j)} - \sum_{m=1}^{i} U_{i-1,k}^{(j)})}{B_{i+1,k}} H_{i+1,k}^{(j)}\). Finally, those \(Q_{ik}^{(j)} - \sum_{m=1}^{i} U_{i-1,k}^{(j)}\) passengers wait until vehicle \(i\) departs stop \(k\), that is, \(\sum_{\ell=i+2}^{i} H_{\ell k}^{(j)}\) additional time.

The following average waiting time is obtained

\[
AW^{(j)}_{ik} = \frac{1}{Q_{ik}} \times \\
\left[ \sum_{m=1}^{i} U_{i-1,k}^{(j)} \left( \frac{u_{i-1,km}^{(j)}}{2B_{mk}^{(j)} + e_{mk}^{(j)}} H_{mk}^{(j)} + \sum_{\ell=m+1}^{i} H_{\ell k}^{(j)} \right) + (Q_{ik}^{(j)} - \sum_{m=1}^{i} U_{i-1,k}^{(j)}) \times \\
\frac{\min\{b_{ik}^{(j)}, u_{i-1,k}^{(j)}\}}{b_{ik}^{(j)} + e_{ik}^{(j)}} H_{ik}^{(j)} \right] \times \\
\left[ \frac{Q_{ik}^{(j)} - \sum_{m=1}^{i} U_{i-1,k}^{(j)}}{B_{i+1,k}} H_{i+1,k}^{(j)} + \sum_{\ell=i+2}^{i} H_{\ell k}^{(j)} \right],
\]

and it coincides with (7). \(\Box\)
Proof of Proposition 3: For every 1 ≤ m ≤ i - 1, all the old passengers of $SU_{l_{i-1,k,j,m}}^{(j)}$ board vehicle $i$ of route $j$ at stop $k_j$ as follows. Let $i_f^{(-j)}(m) ≤ n ≤ i_{l_i}^{(-j)}(m)$, then passengers from $SU_{B_{ik,m,n}}^{(j)}$ board the vehicle first. These passengers arrived to that stop during $h_{mn}^{(j)} - h_{mn}^{(i)}$ minutes (or $h_{mn}^{(j)} - h_{mn}^{(i)}$, if $n = i_f^{(-j)}(m)$), and they first wait half of this duration in average. They also wait since vehicle $n$ of route $(-j)$ arrives until vehicle $m$ of route $j$ departs, i.e., $h_{mn}^{(j)}$ minutes, and then $\sum_{t=m+1}^{i} H_{tk_j}^{(j)}$ additional minutes, until vehicle $i$ of route $j$ departs. The next passengers who board vehicle $i$ are those of $SU_{l_{i-1,k,j,mn}}^{(j)}$. Those passengers first wait since vehicle $n$ of route $(-j)$ arrives until vehicle $m$ of route $j$ departs, namely, $h_{mn}^{(j)}$ minutes. Then, they wait $\sum_{t=m+1}^{i} H_{tk_j}^{(j)}$ additional minutes. The last old passengers who board vehicle $i$ are those of $SU_{B_{ik,m,n}}^{(j)}$. Next, all new passengers from $\bigcup_{n=i_f^{(-j)}}^{i_{l_i}^{(-j)}} B_{ik,m,n}^{(j)}$ as well as $U_{l_{i-1,k_j}}^{0} + \sum_{n=i_f^{(-j)}}^{i_{l_i}^{(-j)}} (B_{ik,m,n}^{(j)} + T_{ik,m,n}^{(j)})$ board the vehicle. Those of $SB_{ik,m,n}^{(j)}$ wait in average half of their arrival duration, $h_{in}^{(j)} - h_{in}^{(i)}$ minutes (or $h_{ik}^{(j)} - h_{ik}^{(i)}$, if $n = i_f^{(-j)}(i)$), and then $h_{in}^{(j)}$ additional minutes, until vehicle $i$ of route $j$ departs (thus, their total average time is $(\min(h_{in}^{(j)} - h_{ik}^{(i)}), h_{in}^{(j)})/2$). Passengers of $ST_{n}^{(-j)}$ wait $h_{in}^{(j)}$ minutes.

Assume that (11) is satisfied, then the last passengers who board vehicle $i$ are additional passengers from a subset of $SB_{ik,j_{i_o+1}}^{(j)}$ board. This subset contains $Q_{ik_j}^{(j)} - \left[ q_{ik_j}^{(j)} - \left( q_{ik_j}^{(j)} + \sum_{n=i_f^{(-j)}}^{i_{l_i}^{(-j)}} \left( B_{ik,m,n}^{(j)} + T_{ik,m,n}^{(j)} \right) \right) \right]$ people, and they arrived to stop $k_j$ before the rest $SB_{ik,j_{i_o+1}}^{(j)} = \left[ Q_{ik_j}^{(j)} - \left( q_{ik_j}^{(j)} + \sum_{n=i_{l_i}^{(-j)}}^{i_{l_i}^{(-j)}} \left( B_{ik,m,n}^{(j)} + T_{ik,m,n}^{(j)} \right) \right) \right]$ did. Thus, they first wait half of their arrival time, $\frac{q_{ik_j}^{(j)} - q_{ik_j}^{(j)} + \sum_{n=i_f^{(-j)}}^{i_{l_i}^{(-j)}} \left( B_{ik,m,n}^{(j)} + T_{ik,m,n}^{(j)} \right)}{2} (h_{iio}^{(j)} - h_{i_{i_o+1}}^{(j)})$ (or $H_{ik_j}^{(j)} - h_{i_{i_o+1}}^{(j)}$, if $i_{i_o} = i_f^{(-j)}(i) - 1$, or $h_{i_{i_o}}$, if $i_{i_o} = i_{l_i}^{(-j)}(i)$). Then, they wait during the arrival time of the rest passengers of $SB_{ik,j_{i_o+1}}^{(j)}$, which is equal to $\frac{q_{ik_j}^{(j)} - q_{ik_j}^{(j)} + \sum_{n=i_{l_i}^{(-j)}}^{i_{l_i}^{(-j)}} \left( B_{ik,m,n}^{(j)} + T_{ik,m,n}^{(j)} \right)}{2} (\min(h_{iio}^{(j)} H_{ik_j}^{(j)}), \max(h_{i_{i_o+1}}, 0))$. Finally, they wait additional $\max(h_{i_{i_o+1}}, 0)$ minutes, until vehicle $i$ of route $j$ departs.

Hence, the average waiting time of the passengers from $SB_{ik,j_{i_o+1}}^{(j)}$ becomes
\[
Q_{ikf}(j) - \left( u_{i-1,kf} + \sum_{n=0}^{l_{(i-1,kf)}} (b_{ikfn} + r_{n}^{(f)}) \right) \frac{\min(h_{l_{(i+1,kf)}}, 0) - \max(h_{l_{(i,kf)}}, 0)}{b_{ikf,k_{l_{(i,kf)}}}} + sb^{(j)}_{ikf,l_{(i+1,kf)}} - \left( u_{i-1,kf} + \sum_{n=0}^{l_{(i-1,kf)}} (b_{ikfn} + r_{n}^{(f)}) \right) \frac{\min(h_{l_{(i,kf)}}, H_{ikf}) - \max(h_{l_{(i+1,kf)}}, 0) + \max(h_{l_{(i,kf)}}, 0)}{b_{ikf,l_{(i+1,kf)}}}.
\]

On the other hand, if (11) is met, then all the new passengers from \( SB_{ikf,l_{(i+1,kf)}} \) board vehicle \( i \), and their average total waiting time is \( \frac{\min(h_{l_{(i,kf)}}, H_{ikf}) - h_{l_{(i+1,kf)}}}{2} + \frac{h_{l_{(i,kf)}}}{2} \), namely, \( \left( \min(h_{l_{(i,kf)}}, H_{ikf}) + h_{l_{(i,kf)}} \right) / 2 \). The last passengers who board vehicle \( i \) at stop \( k_{f} \) are from a subset of \( ST_{l_{(i+1,kf)}}^{(-j)} \), which contains \( Q_{ikf}(j) - \left( u_{i-1,kf} + \sum_{n=0}^{l_{(i-1,kf)}} (b_{ikfn} + r_{n}^{(f)}) \right) + B_{ikf,l_{(i+1,kf)}} \) (out of \( ST_{l_{(i+1,kf)}}^{(-j)} \) in whole \( ST_{l_{(i+1,kf)}}^{(-j)} \)). Their waiting time is equal to \( h_{l_{(i+1,kf)}} \). \( \square \)