Optimal Transit Service Design in a Linear Corridor Connecting Multiple Job Centers Considering Residential Location Choice

Liang Xia[1], Xinguo Jiang[1], Weihua Gu[2], Wenbo Fan[1 *]

Abstract: A continuum approximation (CA) model is formulated in this study for a long-term transit corridor design considering residential location choices. The modeled region is a linear corridor connecting multiple job centers (of which the traditional monocentric model becomes a special case), and households are continuously located along the corridor. The derived travel demand is split to the job centers according to the proximity of their residential locations to the centers. This paper optimizes the transit system design with respect to the combinations of short- and long-haul lines, the location-varying stop density, and service headways under the constraint of housing market equilibrium in a long run. The optimization problem is solved by a two-stage approach integrating the Direct Iteration method and the Lagrange Multiplier Method with the Karush-Kuhn-Tucker condition. Numerical results show that: (i) as opposed to the conventional transit design models (that are mostly built in a monocentric corridor), the proposed model reveals substantially different results regarding to the household distribution and the optimal design of the transit system; (ii) it is found that the optimal transit system is mainly influenced by the distance between centers as well as their attractions (quantified by travel demand split ratios of households); and (iii) different transit modes (e.g., bus and metro) may lead to various long-term development patterns in the corridor in terms of the transit configuration, corridor boundaries, as well as household distribution.

Keywords: Public transit design, linear corridor, housing equilibrium, continuum approximation, long-term planning, multiple job centers

1 Introduction

Serving as the major mass transport mode, transit systems impose substantial impacts on urban land use pattern, property value, and housing market (*Cervero and Landis, 1997; Ho and Wong, 2007; Suzuki et al.,2013; Yin et al., 2013; Kay et al., 2014*). Associated with the changes in urban land use, the long-term transit demand varies and further affects the selection of transit technologies and the optimal configuration of the transit network (*Vuchic, 2005*). The interactions between transit system and land use has long been recognized (*Wegener, 2004; Acheampong and Silva, 2015*). It is therefore expected that an integrated planning of transit system and property development would render modern cities an opportunity of sustainable growth against the deteriorating roadway traffic congestion and air pollution. However, the literature has so far offered a limited guidance to the practice request. Consequently, the performance of the transit network planning in practice may be suboptimal and myopic using the conventional design models that by-and-large ignored or over-simplified the feedback of land use.

The literature on transit network design is plenty. Many studies have explicitly incorporated into their models with substantial real-world details, including different network forms (e.g., grid and ring-radial), multiple operating schemes (e.g., all-stop and skip-stop), and geographical constraints on route and stop locations (*Holroyd, 1967*; *Daganzo, 2010; Freyss et al., 2013; Ouyang, et al., 2014; Chen et al., 2015*). Nonetheless, a majority of the existing transit design models are assumed to have specific transit demand patterns, which are explicitly or implicitly determined by exogenous population distributions (*Daganzo, 2010; Chen et al., 2015; Newell, 1973; Vaughan, 1986*), or are determined by origins and destinations distribution, such as one-to-many, many-to-one and many-to-many (*Ghoneim and Wirasinghe, 1987; Newell 1979; Li et al., 2012*). Several advanced transit design models have been developed with spatially heterogeneous demand to reflect the variations of land use (*Ouyang, et al., 2014*, *Chien and Schonfeld, 1997; Chien and Schonfeld, 2002*). These models, however, are static and cannot capture the interactions between transit system and land use.

Most recently, Li et al. (*2012*) conducted a pioneering work by integrating the rail line design and property development, which accounts for the households' residential location choice behaviours. Furthermore, Li et al. (*2015*) applied the integrated model framework to the transit technology selection and investment timing problems. Several insightful findings were presented in their study, such as the significant impacts of integrating property development on the optimal transit design and the population thresholds of transit technology investment. These two pioneering works, however, have several limitations: for instance, (i) a fixed and uniform housing supply was assumed in Li et al. (*2012*) at any locations in the city, which failed to simulate the property developers' behaviour (e.g., varying housing

---

[*] corresponding author: wbfan@swjtu.edu.cn

[1] School of transportation and logistics, Southwest Jiaotong University, China; [2] Department of Electrical Engineering, The Hong Kong Polytechnic University, China

supply) in response to the design of transit system; (ii) travellers' transit travel cost was simplified in Li et al. (2015) as a linear function of distance irrelevant to any transit configurations, which inevitably restricted the proposed model in guiding the optimal design of the transit system; and (iii) the two studies were limited to monocentric corridors, where all the patrons travel to a single Central Business District (CBD) from home. Empirical evidences show that transit corridors connecting multiple activity centers (typically two, e.g., one CBD and one subcenter, or a pair of neighboring cities) widely exist in the world (Yiu and Tam, 2007; Yiu, 2011). In a polycentric corridor, the housing location problem is very different from the case of monocentric corridor, particularly for households with multiple workers who commute to different activity centers along the corridor. To our best knowledge, the transit design problem considering housing location choices in a polycentric corridor has not been addressed in the literature.

In light of this, the paper develops a continuum approximation model for the transit design in a bi-centric corridor considering the patrons' residential location choice in a long term. Three major extensions to the above-mentioned studies are made: (i) the housing demand-supply equilibrium is explicitly established to simulate both households' location choice behavior and property developers' housing supply behavior; (ii) the optimal design of a corridor transit system is placed in a linear polycentric corridor, which enables a general representation of modern cities with multiple activity centers; and (iii) heterogeneous design (in terms of location-varying stop density) is optimized to the transit system so as to best fit the endogenous non-uniformly distributed demand. Additional insights are shed by applying the proposed model to the transit technology selection problem with regard to the changes in the urban population and the income level. Moreover, the optimal design problem is formulated into a mathematical program minimizing the total system cost (i.e., the sum of households' commute cost and transit agency cost) with housing equilibrium constraints (MPEC). This MPEC is solved by a two-stage approach employing the Direct Iteration Method and the Lagrange Multiplier Method with the Karush-Kuhn-Tucker condition (KKT). Our methodology can be easily extended to model corridors with more than two centers.

The remainder of this paper is organized as follows: next section presents the basic modeling assumptions and the corridor layout. Section 3 presents the transit system metrics, and the housing market equilibrium, followed by the solution algorithm in section 4. Section 5 conducts the numerical studies to demonstrate the effectiveness of the proposed model. Conclusions and future researches are drawn in the last section.

2 Modeling Assumptions and Corridor Layout

Without loss of generality, the following assumptions are made in this paper to facilitate the model development:

A1. The corridor of concern is assumed to be linear; the land within the corridor boundary is a featureless plain identical and ready for the residential use; and the value of land at/beyond the boundary is equal to the agricultural rent or opportunity cost of the land (Fallis, 1985). All employment, shopping, and other activities occur in multiple pre-specified centers (i.e., two in this study) that are not necessarily of identical characteristics, e.g., a primary central business center (CBD) and a secondary CBD, namely Center I and II (as shown in Figure 1) (Romanos, 1977).

A2. The total population of households in the corridor is exogenously given and fixed. All households are assumed to be homogeneous with respect to the socioeconomic characteristics (e.g., value of time). With the aim of maximizing the utilities, households make decisions on the choices of the residential locations and housing consumption (in terms of gross floor area per household) within their budget constraints (Alonso, 1974; Fallis, 1985; Fujita, 1989; Magliocca et al. 2012). Their utility function follows the form of a Cobb-Douglas function (Fallis, 1985), which is additive, separable, and logarithmic.

A3. The property developers determine the intensity of capital investment in the perfectly competitive housing market to maximize their net profits (assumed as zero) from the supply of housing service (Romanos, 1977; Fallis, 1985; Li et al., 2015). The housing supply of property developer is assumed to follow a Cobb-Douglas, first-degree homogeneous function of the inputs of land and capital (Beckmann, 1974; Romanos, 1977; Fujita, 1989; Li et al., 2015).

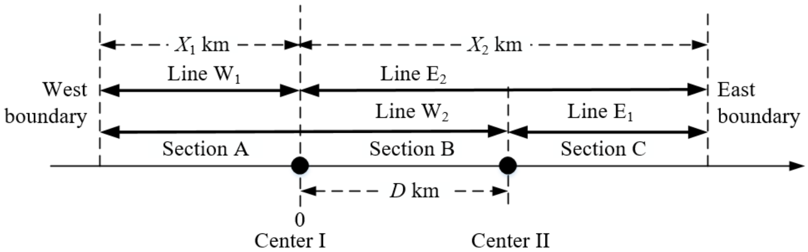


Fig. 1 Transit system in a linear two-center corridor

In the above corridor, the travel demand is derived from households that consist of multiple workers, who commute from their residences to the predetermined locations of employment in the centers. The total number of trips (denoted by σ) per household per day is assumed to be constant, and is split into two centers by $k_I(x),\ k_{II}(x) > 0$ ($k_I(x) + k_{II}(x) = \sigma$), where $k_I(x),\ k_{II}(x)$ depends on the proximity to the Center I and II, respectively (see the Appendix A) (*Evans, 1973*). To serve the daily commuting trips, there are two transit lines in each direction: one of them runs from the boundary to the near center (i.e., the short-haul Line W1 and E1 at west and east sides, respectively), and the other connects the far center (i.e., the long-haul Line W2 and E2, respectively), as shown in Fig. 1. Transit vehicle stops at each stop to pickup and delivery patrons, and the average dwell time at each stop is assumed to be constant (*Wirasinghe, 1981; Kocur and Hendrickson, 1982; Nourbakhsh and Ouyang, 2012; Sivakumaran, et al., (2014)*). In the long run, considering households' location choices, the transit agency seeks to optimize the transit system design to best serve the derived travel demand. Next section presents the model formulation of the optimal design problem.

3 Model Formulations

The transit system design is to determine the stop density ($\rho(x)$, as a function of location x) and the headway (H_{ij}, as scalar variables). The objective is to minimize the generalized system cost (denoted by Z), which is the sum of patrons' commuting cost (Γ) and transit agency's cost (Λ_a). The optimal design problem is formulated as follows.

$$\min Z = \frac{1}{N\mu}\left(\Gamma\left(X_1, X_2, n(x), H_{ij}, \rho(x)\right) + \Lambda_a\left(X_1, X_2, n(x), H_{ij}, \rho(x)\right)\right) \tag{1a}$$

subject to:

Vehicular capacity constraint: $O_{ij}\left(n(x), X_1, X_2, H_{ij}\right) \le C_V$ (1b)

Headway constraint: $H_{i2} = m_i H_{i1}$, m_i=1, 2, 3, …, or 1/2, 1/3, 1/4, … (1c)

Housing market equilibrium: see Eqs. (12a-c) in section 3.3

where N is the total number of households; and μ is the value of time for the transit patrons ($/h). X_1 and X_2 (km) are the west and east boundaries of the corridor. $n(x)$ (households/km) is the household density at location x. In the vehicular capacity constraint (1b), the $O_1(\cdot)$ and $O_2(\cdot)$ are the maximum vehicle occupancy along the transit line operating from west and east sides; and C_V denotes the vehicle's passenger-carrying capacity. Constraint (1c) regulates a dispatching scheme of two transit lines that maintain a regular headway of vehicles at each side of the corridor, of which detailed derivation is furnished in Appendix B. Detailed formulations of $\Gamma(\cdot)$, $\Lambda_a(\cdot)$, $O_1(\cdot)$ and $O_2(\cdot)$ are presented in section 3.1 and 3.2. Lastly, the housing market equilibrium (described in section 3.3) determines the stable household distribution ($n(x)$) in the long run under the optimal design of the transit system.

3.1 Patrons' commuting cost

For commuters travelling from their residential location x to Center i, their travel time, denoted $\varphi(x)$, equals to the sum of three components, i.e., the access time to the closest transit stop, $A(x)$, the waiting time at the transit stop, $W(x)$, and the in-vehicle travel time from the stop to the destination, $T(x)$:

$$\varphi(x) = A(x) + G(x) + T(x), \forall x \in [-X_1, X_2] \tag{2}$$

Continuum approximation is applied to the above three components. First, it is assumed that the access time ($A(x)$) for each commuter within two consecutive stops is represented by the average time of an uniformly distributed population, i.e., $\frac{1}{4v_w\rho(x)}$, which is the average access distance ($\frac{1}{4\rho(x)}$) divided by the walking speed (v_w). Second, the waiting time ($G(x)$) at stops is a function of transit headway $\frac{H_i}{2}$ (Described in Appendix B) or $\frac{H_{i2}}{2}$, as assumed in the previous studies (*Medina et al. 2013*). Third, the estimation of the in-vehicle travel time ($T(\cdot)$) is consisted of delays in stop, $t_d\rho(x)$, which is product of delays in each stop (t_d, caused by acceleration, dwelling and deceleration of transit vehicle at stop), and stop density ($\rho(x)$), and the travel time with cruising speed ($\frac{1}{v_t}$), which v_t is the cruising speed. For the estimation of travel time with cruising speed ($T(\cdot)$), the difference between the time of vehicle travelling to the destination from location x and from the closest transit stop (i.e., $\frac{1}{4v_s\rho(x)}$ on average, where v_s the average operating speed of transit vehicles) is ignored so as to avoid the discrete nature associated with the integer number of stop spacing in the latter estimation. Thus, the $T(\cdot)$ is approximately expressed as a continuous function of ($t_d\rho(x) + \frac{1}{v_t}$) from the commuters' residential location (x) to the Center I and II, i.e., $\left|\int_x^0 \left(t_d\rho(x) + \frac{1}{v_t}\right)dx\right|$ and $\left|\int_x^D \left(t_d\rho(x) + \frac{1}{v_t}\right)dx\right|$. Note that the above approximation of the in-vehicle travel time is acceptably accurate: for instance, the estimation error ($\frac{1}{4v_s\rho(x)}$)

would be less than 0.5 minutes for rail transit with the average operating speed of 60 km/h and the stop spacing in the range of [0, 2] km. Thus, Eq. (2) can expressed by:

$$\varphi_I(x) = \begin{cases} \frac{1}{4v_w\rho(x)} + \frac{H_W}{2} + \int_x^0 \left(t_d\rho(x) + \frac{1}{v_t}\right)dx, \forall x \in [-X_1, 0] \\ \frac{1}{4v_w\rho(x)} + \frac{H_{W1}}{2} + \int_0^x \left(t_d\rho(x) + \frac{1}{v_t}\right)dx, \forall x \in [0, X_2] \end{cases} \tag{3a}$$

$$\varphi_{II}(x) = \begin{cases} \frac{1}{4v_w\rho(x)} + \frac{H_{E1}}{2} + \int_x^D \left(t_d\rho(x) + \frac{1}{v_t}\right)dx, \forall x \in [-X_1, D] \\ \frac{1}{4v_w\rho(x)} + \frac{H_E}{2} + \int_D^x \left(t_d\rho(x) + \frac{1}{v_t}\right)dx, \forall x \in [D, X_2] \end{cases} \tag{3b}$$

Where D is the given distance between Centers I and II. So, the total daily travel cost (Γ) is the integral of each commuting cost, $\mu\varphi(x)$, along the corridor.

$$\Gamma = \mu \int_{-X_1}^{X_2} (A(x) + G(x) + T(x))n(x)dx \tag{4a}$$

The total daily access cost (Γ_a), daily waiting cost (Γ_w), and daily in-vehicle travel cost (Γ_T) for the entire population are introduced to simplify expression, and they are the integral of the each access cost ($\Gamma_a = \mu \int_{-X_1}^{X_2} A(x)n(x)dx$), the each waiting cost ($\Gamma_G = \mu \int_{-X_1}^{X_2} G(x)n(x)dx$); the each in-vehicle cost ($\Gamma_T = \mu \int_{-X_1}^{X_2} T(x)n(x)dx$), along the corridor. Then, the total daily travel cost can be expressed by:

$$\Gamma = \Gamma_A + \Gamma_G + \Gamma_T \tag{4b}$$

where $\Gamma_A, \Gamma_G, \Gamma_T$ are formulated as follows:

$$\Gamma_A = \mu \int_{-X_1}^{X_2} \frac{\sigma n(x)}{4v_w\rho(x)}dx \tag{5}$$

$$\Gamma_G = \mu \left(\int_{-X_1}^{0} \frac{H_W}{2} k_I(x)n(x)dx + \int_{-X_1}^{D} \frac{H_{W2}}{2} k_{II}(x)n(x)dx + \int_0^{X_2} \frac{H_{E2}}{2} k_I(x)n(x)dx + \int_D^{X_2} \frac{H_E}{2} k_{II}(x)n(x)dx \right) \tag{6}$$

$$\Gamma_T = \mu \int_{-X_1}^{X_2} B(x) \left(\frac{1}{v_t} + t_d\rho(x) \right) dx \tag{7}$$

where $B(x)$ is the cumulative in-vehicle passengers in two directions, which is formulated as:

$$B(x) = \begin{cases} \int_{-X_1}^{x} \sigma n(x)dx \quad x \in [-X_1, 0) \\ \int_{-X_1}^{x} k_{II}(x)n(x)dx + \int_x^{X_2} k_I(x)n(x)dx \quad x \in [0, D] \\ \int_x^{X_2} \sigma n(x)dx \quad x \in (D, X_2] \end{cases} \tag{8}$$

The critical vehicle occupancy is determined by the maximum value observed at the left and right sides of the two centers:

$$O_{W1} = H_{L1}\kappa \cdot \int_{-X_1}^{0} f_{L1}k_I(x)n(x)dx \tag{9a}$$

$$O_{W2} = H_{W2}\kappa \cdot \max\left(\int_{-X_1}^{D} k_{II}(x)n(x)dx, \int_{-X_1}^{0} f_{W2}k_I(x)n(x)dx + \int_{-X_1}^{0} k_{II}(x)n(x)dx \right) \tag{9b}$$

$$O_{E1} = H_{E1}\kappa \cdot \int_D^{X_2} f_{E1}k_{II}(x)n(x)dx \tag{9c}$$

$$O_{E2} = H_{E2}\kappa \cdot \max\left(\int_0^{X_2} k_I(x)n(x)dx, \int_D^{X_2} f_{E2}k_{II}(x)n(x)dx + \int_D^{X_2} k_I(x)n(x)dx \right) \tag{9d}$$

Where κ is the design hour factor, defined as the ratio between peak hour demand and daily demand; f_{ij} is the frequency from boundary *i* to nearer center by line *j*, which is described in Appendix B.

3.2 Transit agency's cost

The transit agency cost depends upon four metrics (*Daganzo, 2010*): the length of transit line (L), the number of transit stops (S), the transit vehicle kilometres travelled per day (K), and the transit vehicle hours travelled per day (M). So, the agency cost is given by:

$$\Lambda_a = \pi_L L + \pi_s S + \pi_K K + \pi_M M \tag{10}$$

where $\pi_L, \pi_S, \pi_K, \pi_M$ are the unit costs related to L, S, K, M, respectively. The four-cost metrics are formulated as follows:

$$L = 2(X_1 + X_2) \tag{11a}$$

$$S = \tau \int_{-X_1}^{X_2} \rho(x)dx \tag{11b}$$

$$K = 2T_r \left(\frac{X_1}{H_{W1}} + \frac{(X_1 + D)}{H_{W2}} + \frac{X_2}{H_{E2}} + \frac{(X_1 - D)}{H_{R1}} \right) \tag{11c}$$

$$M = \frac{K}{v_t} + \tau t_d T_r \left(\frac{\int_{-X_1}^{0} \rho(x)dx}{H_{W1}} + \frac{\int_{-X_1}^{D} \rho(x)dx}{H_{W2}} + \frac{\int_{0}^{X_2} \rho(x)dx}{H_{E2}} + \frac{\int_{D}^{X_2} \rho(x)dx)}{H_{E1}} \right) \quad (11d)$$

where $X_1 + X_2$ is the line length in one direction; 2 expresses two directions in Eq. (11a); τ is the number of transit stops in two directions in Eq. (11b). T_r is the transit daily operation hours, and $\frac{X_1}{H_{W1}}$, $\frac{(X_1+D)}{H_{W2}}$, $\frac{X_2}{H_{E2}}$ and $\frac{(X_1-D)}{H_{E1}}$ are the trip distance of four lines per hour in Eq. (11c), respectively. In Eq. (11d) $\frac{K}{v_t}$ is the operation time of vehicle with desire velocity in four lines per hour; $\frac{\int_{-X_1}^{0} \rho(x)dx}{H_{W1}}$, $\frac{\int_{-X_1}^{D} \rho(x)dx}{H_{W2}}$, $\frac{\int_{0}^{X_2} \rho(x)dx}{H_{E2}}$ and $\frac{\int_{D}^{X_2} \rho(x)dx)}{H_{E1}}$ are the stop number of the vehicles passed in four lines per hours, respectively.

3.3 Housing market equilibrium

In the two-center corridor, the housing market spatial equilibrium satisfies three conditions with respect to the accommodated population and land value at the corridor boundaries. Detailed derivations are furnished in Appendix C and D. Here we directly summarize the final results as follows. First, the total population of households, N, fits exactly inside the corridor area defined by its boundaries (X_1, X_2), expressed by:

$$\int_{-X_1}^{0} n(x, p_0)dx + \int_{0}^{D} n(x, p_0)dx + \int_{D}^{X_2} n(x, p_0)dx = N \quad (12a)$$

where X_1, X_2 is the to-be-determined distance from the corridor boundary to the Center I; where p_0 represents the average housing rental price at Center I.

Second, according to A1, the land value at the corridor boundaries equals the exogenous agriculture rent, and thus the following two equations hold true:

$$r(-X_1, p_0) = b^{\frac{-\beta}{1-\beta}}(\beta^{-1} - 1)\left(p_0 \left(\frac{Y-\psi(-X_1)}{Y-\psi(0)}\right)^{\frac{1}{(1-\alpha)}} \gamma\beta\right)^{\frac{1}{1-\beta}} = r_a \quad (12b)$$

$$r(X_2, p_0) = b^{\frac{-\beta}{1-\beta}}(\beta^{-1} - 1)\left(p_0 \left(\frac{Y-\psi(X_2)}{Y-\psi(0)}\right)^{\frac{1}{(1-\alpha)}} \gamma\beta\right)^{\frac{1}{1-\beta}} = r_a \quad (12c)$$

where r_a is a constant agricultural rent; Y is household's income per year; $\psi(x)$ is the annual travel cost per household in the two-center city and is expressed in Eq.(20c); b represents the unit cost of capital. α, β and γ are parameters of households and property developers' utility functions.

Note that on the basis of Eqs. (12a-12c), once the annual travel cost $\psi(x)$ is known at any location x, the housing market equilibrium can be resolved with three independent variables, p_0, X_1, X_2, and subsequently the housing distribution, $n(x, p_0)$, can be determined.

4 Solution Method

The optimization model described in section 3 includes an integer constraint. To improve the computational efficiency, we first solve a relaxed program without the integer constraint (1c). We then search for the lowest-cost solution that satisfies (1c) in the neighbourhood of the relaxed program's optimal solution (*Fan et al., 2018*). Moreover, as described in section 3, the model is split to three steps: trip generation, trip distribution, transit optimization, which can be summarized as demand spatial distribution and transit optimization, as shown in Fig. 2. The demand spatial distribution parameters, $n(x)$, X_1, X_2, $k_{\mathrm{I}}(x)$ and $k_{\mathrm{II}}(x)$, determine transit system optimization. At the same time, the transit metrics, $\rho(x)$, H_1 and H_2, also influence the demand spatial distribution by travel time function $\psi(x)$ in trip generation. The relationship between the trip generation parameters, $n(x)$, X_1 and X_2, and the trip distribution parameters, $k_I(x)$ and $k_{II}(x)$, are unidirectional in demand spatial distribution. So, there is a bi-level program consisting of the demand spatial distribution and transit optimization. Moreover, when the demand spatial distribution and mode split are solved, the transit optimization can be solved by Lagrange Multiplier Method with the Karush-Kuhn-Tucker (KKT) conditions, and the results are as follows:

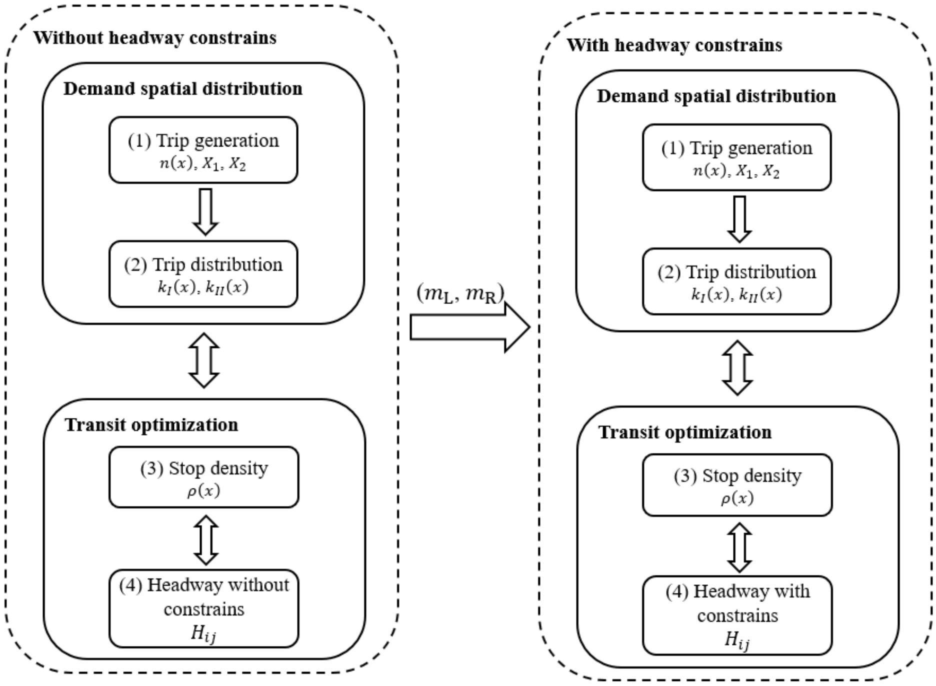


Fig. 2 Flow chart of solution method

The optimal conditions of density $\rho(x)$ could be expressed as:

$$\rho^*(x) = \frac{1}{2}\sqrt{\frac{\mu\sigma n(x)}{v_W(\mu t_d B(x)+\pi_M T_r t_d \boldsymbol{H'}\tau+\pi_S\tau)}} \tag{13a}$$

Where,

$$H' = \begin{cases} \dfrac{1}{H_{W1}}+\dfrac{1}{H_{W2}} & x \in [-X_1, 0) \\ \dfrac{1}{H_{W2}}+\dfrac{1}{H_{E2}} & x \in [0, D] \\ \dfrac{1}{H_{E1}}+\dfrac{1}{H_{E2}} & x \in (D, X_1] \end{cases} \tag{13b}$$

The optimal conditions of headway H_{ij} without headway constrain could be expressed as:

$$H_{W1}^* = \frac{H_{W2}}{\sqrt{\frac{\mu v_t H_{W2}^2\left(\int_{-X_1}^{0} k_I(x)n(x)dx\right)}{T_r\left(\left(4(\pi_K v_t+\pi_M)X_1+2\pi_M v_t\tau t_d\left(\int_{-X_1}^{0}\rho(x)dx\right)\right)\right.}}-1} \tag{14a}$$

$$\left(\frac{H_{W1}}{\boldsymbol{H_{W2}^*}+H_{W1}}\right)^2 \mu\int_{-X_1}^{0} k_I(x)n(x)dx + \mu\int_{-X_1}^{\mathrm{D}} k_{II}(x)n(x)dx = \frac{T_r\left(4(\pi_K v_t+\pi_M)(X_1+D)+2\pi_M v_t\tau t_d\int_{-X_1}^{\mathrm{D}}\boldsymbol{\rho(x)}dx\right)}{v_t\boldsymbol{H_{W2}^*}} \tag{14b}$$

$$H_{E1}^* = \frac{H_{E2}}{\mathbf{1}-\sqrt{\frac{T_r\left(\left(4(\pi_K v_t+\pi_M)(X_2-D)+2\pi_M v_t\tau t_d\left(\int_{D}^{X_2}\rho(x)dx\right)\right)\right)}{\mu v_t H_{E2}^2\left(\int_{\mathrm{D}}^{X_2} k_{II}(x)n(x)dx\right)}}} - H_{E2} \tag{14c}$$

$$\left(\frac{H_{E1}}{\boldsymbol{H_{E2}^*}+H_{E1}}\right)^2 \mu\int_{\mathrm{D}}^{X_2} k_{II}(x)n(x)dx + \mu\int_{0}^{X_2} k_I(x)n(x)dx = \frac{T_r\left(4(\pi_K v_t+\pi_M)X_2+2\pi_M v_t\tau t_d\int_{0}^{X_2}\boldsymbol{\rho(x)}dx\right)}{v_t\boldsymbol{H_{E2}^*}} \tag{14d}$$

Considering the headway constraint, (14a-d) are rewritten into:

$$H_{W1}^* = \sqrt{\frac{T_r\left((4(\pi_K v_t+\pi_M)((m_W+1)X_1+D)+2\pi_M v_t\tau t_d\left(m_W\int_{-X_1}^{0}\rho(x)dx+\int_{-X_1}^{\mathrm{D}}\boldsymbol{\rho(x)}dx\right)\right)}{\mu v_t m_W\left(\min(1,m_W)\int_{-X_1}^{0} k_I(x)n(x)dx+m_W\int_{-X_1}^{\mathrm{D}} k_{II}(x)n(x)dx\right)}} \tag{15a}$$

$$H_{E1}^* = \sqrt{\frac{T_r\left((4(\pi_K v_t+\pi_M)((m_E+1)X_2-m_R D)+2\pi_M v_t\tau t_d\left(\int_{0}^{X_2}\rho(x)dx+m_E\int_{D}^{X_2}\rho(x)dx\right)\right)}{\mu v_t m_E\left(m_E\int_{0}^{X_2} k_I(x)n(x)dx+\min(1,m_E)\int_{D}^{X_2} k_{II}(x)n(x)dx\right)}} \tag{15b}$$

$$H_{W2}^* = m_W H_{W1}^* \tag{15c}$$

$$H_{E1}^* = m_E H_{E2}^* \tag{15d}$$

As described in Eqs. (13a-15d); the stop density, $\rho(x)$, and the headways, H_{ij}, are interdependent. So, the nonlinear optimization problem can be solved by the nest two-stage procedure employing the Direct Iteration Method. The step-by-step procedure is summarized as follows:

Step 0 Initialization. Assign initial values to the decision variables $k_I^{(0)}(x)$, $k_{II}^{(0)}(x)$, $p_0^{(0)}$, $X_1^{(0)}$, $X_2^{(0)}$, $\rho^{(0)}(x)$, $H_{ij}^{(0)}$. Set the iteration counter $\eta_1 = 1$. If *flag* is not exit, set *flag*=0.

Step 1 Solve the demand spatial distribution.

Step 1.1 Solve the trip generation.

Step 1.1.1 Calculate households' annual travel cost $\psi^{(\eta_1)}(x)$ with daily trips, $k_I^{(\eta_1-1)}(x)$, $k_{II}^{(\eta_1-1)}(x)$, and transit meters, $\rho^{(\eta_1-1)}(x)$, and $H_{ij}^{(\eta_1-1)}$, by Eqs. (20c, 3a-3b).

Step 1.1.2 Substitute $\psi^{(\eta_1)}(x)$ to the nonlinear equations (12a-12c) and solve the housing market equilibrium to yield the results of $p_0^{(\eta_1)}, X_1^{(\eta_1)}, X_2^{(\eta_1)}$.

Step 1.1.3 Output the housing density, $n^{(\eta_1)}(x)$, by Eqs. (31, 30, 26).

Step 1.2 Solve the trip distribution

Calculate the daily trips, $k_I^{(\eta_1)}(x)$, $k_{II}^{(\eta_1)}(x)$ at any location $x(x \in [X_1^{(\eta_1)}, X_2^{(\eta_1)}])$, by Eqs. (17a-17b).

Step 2 Solve the transit optimization

Step 2.0 Assign the initial headway $H_{ij}^{(\eta_1,0)}$. Setting the iteration counter $n_2 = 1$.

Step 2.1 Calculate the density of stop $\rho^{(\eta_1,\eta_2)}(x)$ at any location $x(x \in [X_1^{(\eta_1)}, X_2^{(\eta_1)}])$ with the $H_{ij}^{(\eta_1,0)}$ and Eqs. (13a-13b).

Step 2.2 if flag=0, go to Step2.2.1; else go to the Step 2.2.2

Step 2.2.1 Calculate headway without headway constrains. Substitute $\rho^{(\eta_1,\eta_2)}(x)$ to Eqs. (14a-14d) and get the new headway $H_{ij}^{(\eta_1,\eta_2)}$. Then go to the step 2.3

Step 2.2.1 Calculate headway with headway constrains. Substitute $\rho^{(\eta_1,\eta_2)}(x)$ and m_i to Eqs. (15a-15d) and get the new headway $H_{ij}^{(\eta_1,\eta_2)}$.

Step 2.3 Check the convergence. If the relative gap $\frac{|\rho^{(\eta_1,\eta_2)}(x)-\rho^{(\eta_1,\eta_2-1)}(x)|}{\rho^{(\eta_1,\eta_2-1)}(x)}$ in any location x is smaller than a pre-specified tolerance, e.g., ε=0.001, updating the transit metrics, $\rho^{(\eta_1)}(x) = \rho^{(\eta_1,\eta_2)}(x)$, $H_{ij}^{(\eta_1)} = H_{ij}^{(\eta_1,\eta_2)}$, and go to next step; otherwise, set η_2=η_2+1, and go to Step 2.1 at Step 2.

Step 3 Check the convergence.

Step 3.1 If the relative gap $\frac{|n^{(\eta_1)}(x)-n^{(\eta_1-1)}(x)|}{n^{(\eta_1)}(x)}$ in any location x is smaller than a pre-specified tolerance, e.g., ε=0.001, go to the Step 3.2; otherwise, set η_1= η_1+1, and go to Step 1.

Step 3.2 Judgment whether the *flag* is 1, if it is, report the solutions and stop, otherwise, set the *flag* = 1， and go to next step.

Step 3.3 Calculate the set of m_i. If $H_{W2}^{(\eta_1)} > H_{W1}^{(\eta_1)}$, the set of m_W is consisted of the floor and ceiling of $\frac{H_{W2}^{(\eta_1)}}{H_{W1}^{(\eta_1)}}$; else, the set of m_W is consisted of the inverse of the floor and ceiling of $\frac{H_{W1}^{(\eta_1)}}{H_{W2}^{(\eta_1)}}$. It's similar to m_E. Then, repeat Step 0 to Step 2 with four possible combinations of the m_W and m_E, respectively, get the lowest-cost solution among them and stop.

5 Numerical Experiments

We apply the proposed models and solution method to conduct numerical experiments for various operating scenarios. Section 5.1 examines the optimal design of a rail line in a bi-center corridor; section 5.2 compares the rail line designs for monocentric and bi-centric corridors; section 5.3 compares the optimal designs of rail, Bus Rapid Transit (BRT) and regular bus systems; and section 5.4 compares the optimal transit line combinations. The parameter values are furnished in Table 1 (*Romanos, 1977; Gu et al., 2016; Daganzo, 2010*).

Table 1 Parameter values for the numerical experiments

Parameters	Values	Parameters	Values
α	0.75	γ	0.08
β	0.7	D, km	5
w, days/year	250	T_r, h/day	16
μ, \$/h	20	σ	4
b, /year	5%	ξ	0.75
N, households	10,000	δ	0.4

Parameters	Values	Parameters	Values
r_a, \$/km^2/year	3,000	κ	0.1
Parameters	**Rail system**	**BRT system**	**Bus system**
τ, stops	2	1	1
v_w, km/h	2	2	2
v_t, km/h	60	40	25
t_d, s/stop	45	30	30
C_V, spaces/vehicle	1800	100	80
π_L, \$/km/day	14256+475.2 μ	$3888 + 129.6\mu$	144+4.8 μ
π_S, \$/stop/day	7056+235.2 μ	100.8+3.36 μ	10.08+0.336 μ
π_K, \$/veh-km	2.2 μ	0.66 μ	0.59 μ
π_M, \$/veh	2424+5 μ	91.44+4 μ	63.84+3μ

5.1 Optimal rail line design considering two activity centers

Table 2 summarizes the optimal design variables of the rail line with different centers' distance, D={5, 20}km, including the headway and corridor boundaries, as well as the optimal commuting cost, transit agency cost, and total system cost with different centers' distance. The corridor with 20 km distance between two centers results in larger commuting cost (increasing 8.0%) and agency cost (40.1%) than that with only 5 km, but smaller distance between boundary and nearer center. Moreover, larger centers' distance does not only increase the headway to far center, and decrease that to near center, but also brings out a new transit line from right boundary to Center II occurrences.

Fig. 3 and Fig. 4 plot the optimal stop density $\rho(x)$ against x and the locations of every stop (obtained using a discretization recipe similar to *Wirasinghe and Ghoneim, 1981*). The stop density $\rho(x)$ is broken near centers, caused by discontinuous travel attractiveness, which presents as piecewise function of the cumulative in-vehicle passengers $B(x)$. When the centers' distance is 5 km, the higher stop density spots appear inside section B around each center because of smaller cumulative in-vehicle passengers and integrated headway; the highest stop density is observed near Center I because of its higher attraction. However, when the centers' distance is 20 km, the smaller stop density spots appear inside section B and the difference between two higher stop densities near centers is small, that is because the higher cost of the line from farer boundary lead to larger headway from farer boundary and smaller integrated headway from nearer boundary, and the influence of center attraction almost weak to zero. Moreover, in section B, the stop density declines, even to zero when the centers' distance is large enough (such as two-city corridor presented in Fig 4), as the location moves away from the activity centers, same as that in sections A and C. This result is aligned with the household density at the market equilibrium, as shown by the red solid curve in Fig. 5 and Fig. 6. Fig. 5 and Fig. 6 also plot the floor space consumed per household as the blue dashed curve, and one can see that this floor space per household exhibits an opposite trend to the household density as x moves away from the centers. This is also as expected. This numerical example manifests the capability of our models for handling the spatial heterogeneities in transit system design and household distribution jointly.

Table 2 Optimal rail line design with different centers' distance

Design variables and cost terms	D=5 km	D=20 km
Range of stop density, stops/km	(0, 0.64)	(0,0.57)
$\boldsymbol{H}$**, min**	[38.44, 9.61, 7.50, -]	[25.68,12.88,12.15,24.29]
$\boldsymbol{X_1}$**, km**	12.67	10.82
$\boldsymbol{X_2}$**, km**	16.54	30.96
Household's commuting cost, h/ho/day	1.62	1.75
Transit agency cost, h/ho/day	8.01	11.22
Total system cost, h/ho/day	9.63	12.97

The "-" presents the value is more than 60, which means this transit line is not existing.

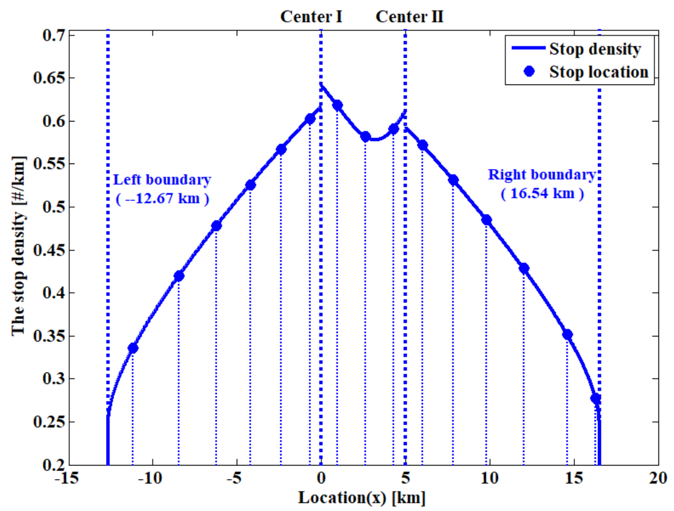


Fig. 3 Optimal rail station density and station locations when centers' distance is 5 km

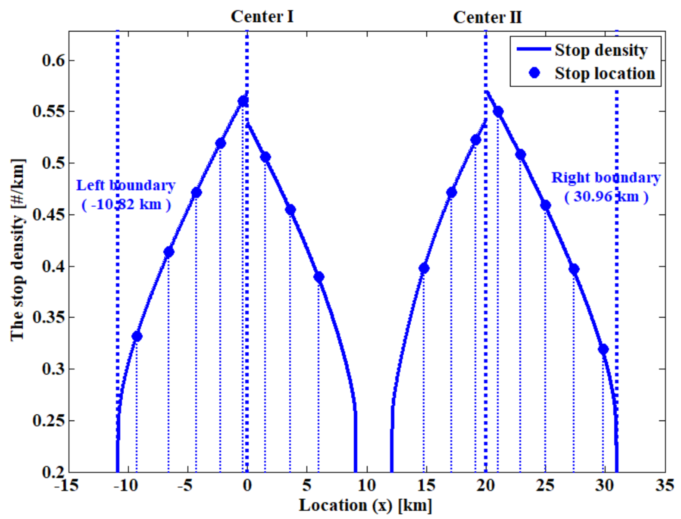


Fig. 4 Optimal rail station density and station locations when centers' distance is 20 km

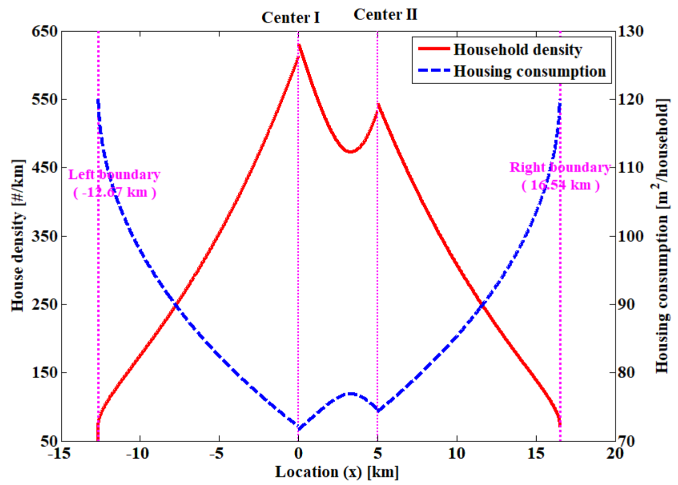


Fig. 5 Optimal household distribution when centers' distance is 5 km

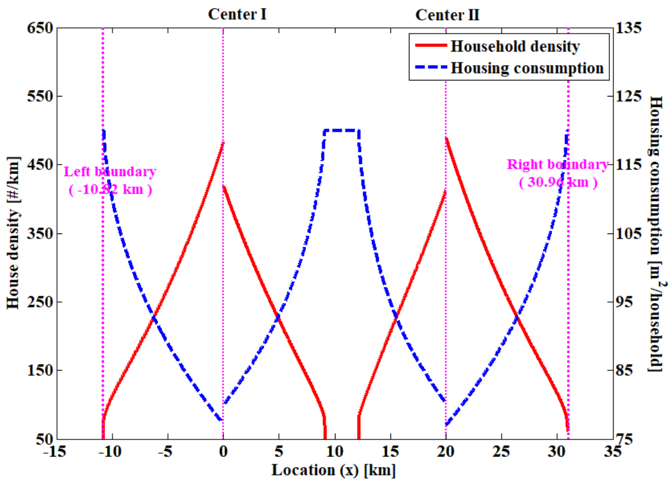


Fig. 6 Optimal household distribution when centers' distance is 20 km

5.2 Comparison between monocentric and bi-centric corridors

We compare three hypothetical corridors: a monocentric corridor with $\xi = 1$; and two bi-centric corridors with ξ=0.75 (asymmetric) and ξ=0.5 (symmetric), respectively. Table 3 presents the optimal designs, Fig. 7 plots the optimal household density distributions for the three corridors, and Fig. 8 plots the optimal stop density distributions for the three corridors. Not surprisingly, the monocentric corridor has the lowest user and agency costs since greater aggregation of activities leads to more cost-effective mass transit system design. However, bi-centric corridors have lower household densities and stop density, especially in the vicinity of the centers where the density is the highest. This implies that residents in a polycentric corridor enjoy more spacious homes, which contributes significantly to the quality of life. Moreover, with the increasing of attraction to Center II, the headway of transit line from left boundary to Center I increases, even it is larger than one hour (presented by '-'). To the contrary, that from left boundary to Center II decreases, even it is smaller than one hour. So, as the attraction changes, the different transit line combinations are presented. Besides, these bi-centric corridors may exist due to geographic, historical, and cultural reasons (e.g., the Hong Kong-Shenzhen rail corridor and the Johor Bahru-to-Singapore rapid transit corridor examined in *Yiu, 2011*).

Table 3 Optimal designs for the monocentric and bi-centric corridors

Design variables and cost terms	Monocentric	Bi-centric	
	$\xi = 1$	$\xi = 0.75$	$\xi = 0.5$
Range of stop density, stop/km	(0, 0.65)	(0, 0.64)	(0, 0.64)
$\boldsymbol{H}$**, min**	[7.05, -, 7.05, -]	[38.44, 9.61, 7.50, -]	[-, 7.80, 7.80, -]
$[-X_1, X_2]$**, km**	[-14.00, 14.00]	[12.67, 16.54]	[12.13, 17.13]
Household's commuting cost, h/ho/day	1.61	1.62	1.63
Transit agency cost, h/ho/day	7.67	8.01	8.02
Total system cost, h/ho/day	9.28	9.63	9.65

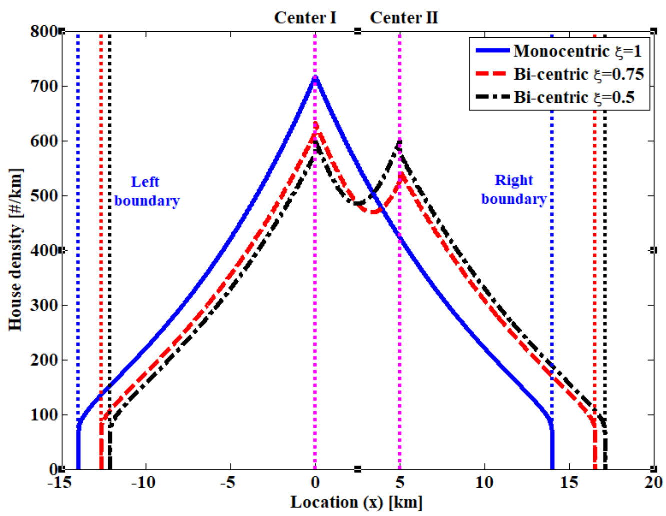


Fig. 7 Household distributions for monocentric and bi-centric corridors

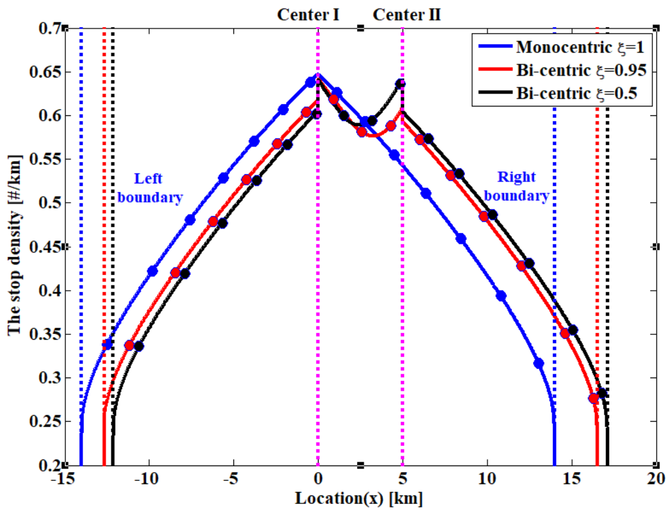


Fig. 8 Station distributions for monocentric and bi-centric corridors

5.3 Comparison between different transit modes

In this section, we compare the optimal rail corridor against the optimal designs of BRT and bus corridors, where all the other operation conditions and parameters are kept the same. Table 4 presents the optimal designs and cost terms for the three transit modes, and Fig. 5 plots the optimal household density distributions. The comparison unveils that, although a bus corridor has the lowest system cost due to the low agency cost, it renders the greatest concentration of households near Center I. On the other hand, rail renders a "flatter" corridor in terms of the household density. BRT has the largest service range (the distance between the left and right boundaries) because it is much cheaper than rail, and much faster than ordinary bus, and hence can better serve the patrons residing in remote suburban areas. Additionally, lower operation cost of BRT and Bus result in higher headway from left boundary to nearer center, that is shown by that this line is not exist.

Table 4 Optimal designs for different transit modes

Design variables and cost terms	Rail	BRT	Bus
Range of stop density, stop/km	(0, 0.64)	(0, 1.69)	(0, 1.87)
***H*, min**	[38.44, 9.61, 7.50, -]	[-,3.56, 4.28, -]	[-, 2.97, 3.84, -]
$[X_1, X_2]$, km	[12.67, 16.54]	[29.81, 33.83]	[21.03,25.46]
Household's commuting cost, h/ho/day	1.62	1.42	1.37
Transit agency cost, h/ho/day	8.01	4.65	0.58
Total transit system cost, h/ho/day	9.63	6.07	1.95

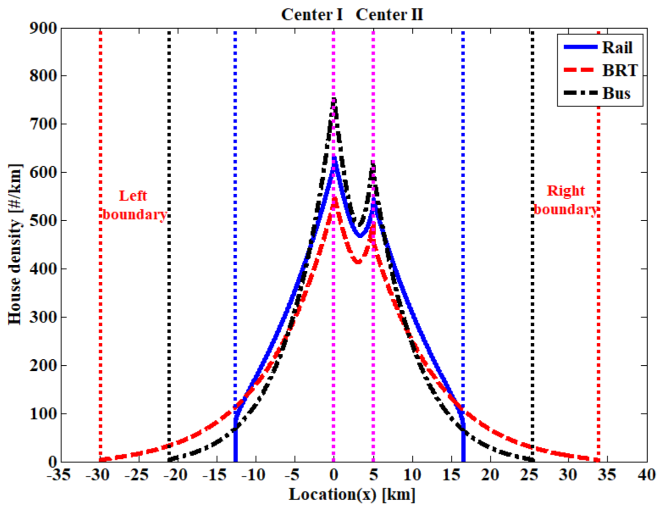


Fig. 9 Household distributions for different transit modes

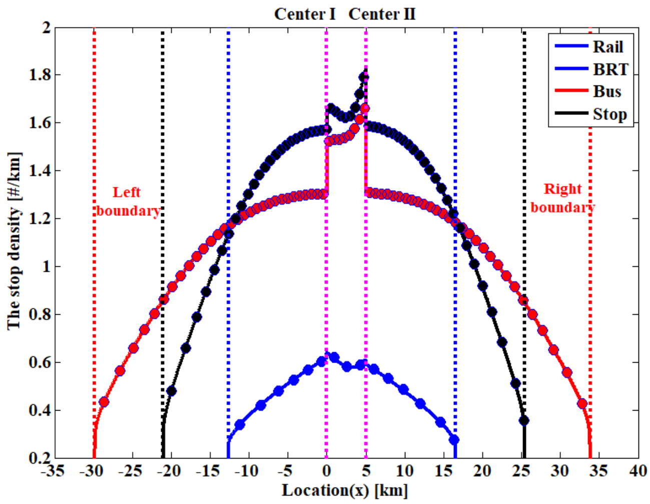


Fig. 10 Station distributions for different transit modes

5.4 Optimal combination of transit lines

In the light of varying transit-lines combination, sensitivity analysis is conducted with respect to the households $N \in [1,10] \cdot 10^4$ and wages $\mu \in [10,50]$. Table 5 and Table 6 present the Optimal transit line combination by the set of four transit lines, $\{W_1, W_2, E_1, E_2\}$. When the W_j / E_j is 1, the headway of this line is smaller than 1 hour, and this line is considered existing; otherwise, this line is not exiting. Those tables show that the center attraction is the main factor influencing the transit-lines combination, the household and the time value affect the combination when the center attraction near 0.3 and 0.6. The combination with one longer line and one short line occurs when there is only one center; the combination with two longer lines occurs when the attraction near 0.5; the combination with four lines is not exist.

Table 5 Optimal transit-lines combinations with difference households (N) and center attractions (ξ)

N (10^4) \ ξ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
1	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
2	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
3	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
4	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
5	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
6	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
7	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
8	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
9	0101	0111	0111	0110	0110	0110	0110	0110	1110	1110	1010
10	0101	0111	0111	0110	0110	0110	0110	0110	1110	1110	1010

Table 6 Optimal transit-lines combinations with difference time values (μ) and center attractions (ξ)

μ (\$/h) \ ξ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
10	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
15	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
20	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
25	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
30	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010
35	0101	0111	0111	0110	0110	0110	0110	0110	1110	1110	1010
40	0101	0111	0111	0110	0110	0110	0110	0110	1110	1110	1010
45	0101	0111	0111	0110	0110	0110	0110	0110	1110	1110	1010
50	0101	0111	0111	0111	0110	0110	0110	1110	1110	1110	1010

6 Conclusion and Future directions

In this paper, a continuum approximation approach was proposed to model the transit design problem in a bi-centric corridor considering households' residential location choices in a long run. The total system cost (i.e., the sum of households' commuting cost and transit agency' cost) is minimized to most effectively serve the travel demand. As opposed to the conventional transit design models (that were mostly built in monocentric corridor with exogenously given or over simplified demand), the proposed model furnishes the optimal transit system for multi-worker households commuting to different job centers.

Some important findings and new insights were obtained. First, the bi-centric model results in substantially different results from the monocentric ones with respect to the population distribution and the optimal configuration of transit system. Second, the inherent characteristics of the corridor, e.g., the distance between two centers and their trip attractions, play a vital role in determining the optimal transit systems (e.g., the combination of short- and long-haul transit lines to two centers). Third, different transit modes may also influence the long-term development of the integrated transit and housing systems. Finally, upon research interests, the proposed bi-centric model could be readily extended to multi-center ones by introducing the additional centers and correspondingly indicating the daily trip attractions to households.

In the future, several extensions can be conducted to the current study: (i) introducing heterogeneous households in terms of their trip patterns, values of time, and annual incomes; (ii) modeling other travel mode in addition to transit, such as private car and public bicycle; and (iii) incorporating a non-monocentric corridor considering companies' location choice and households' destination choices in travel demand model. Selected study is undergoing.

Acknowledgments

The research was supported by the National Natural Science Foundation of China (Project No. 71771191 and 51608455) and a General Research Fund (Project No. 15280116) provided by the Research Grants Council of Hong Kong.

Appendix A Trip distribution

As described in section 2, a gravity distribution model, proposed by Voorhees A.M. (*1956*), is applied in trip distribution. So, the trips of per household and per day to Center i, $k_i(x)$, can be expressed by:

$$k_i(x) = \frac{\mathrm{A}_i f(R_i(x))}{\sum_{j=I}^{II} \mathrm{A}_j f(R_j(x))}\sigma \tag{16}$$

Where A_i is the attraction trips in center i, and they can expressed by the attracted factor ξ, i.e., $\mathrm{A}_I = \xi N\sigma$, and $\mathrm{A}_{II} = (1-\varepsilon)N\sigma$; $R_i(x)$ is the distance from location x to center i, and can expressed by $R_I(x) = |x|$ and $R_{II}(x) = |x - D|$; $f(R_i(x))$ is the distribution cost function, and it can be expressed by $f(R_i(x)) = e^{-\delta R_i(x)}$. So, the $k_I(x)$ can be transformed to:

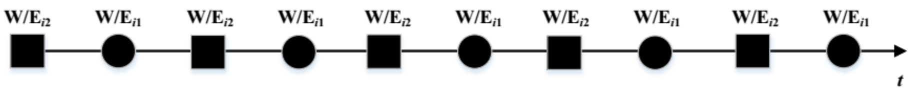
$$k_I(x) = \frac{\xi \mathrm{e}^{-\delta|x|}}{\xi \mathrm{e}^{-\delta|x|} + (1-\xi)\mathrm{e}^{-\delta|x-D|}}\sigma \tag{17a}$$

Meanwhile, the number of trips to Center II, $k_{II}(x)$, can be expressed by:

$$k_{II}(x) = \sigma - k_I(x) \tag{17b}$$

Appendix B Timetable constraints

As described in section 2, there are two transit lines are severed from the boundary to nearer center in one direction. However, the integrated waiting time is not only related to the value of headways, but also are influenced by the headway phases of two lines, i.e., the time interval of the vehicles of two lines. For example, as shown in Fig. 11, there are two types of timetable when the headways are same, under which the average waiting time would be $0.25H_{i1}$ and $0.5H_{i1}$, respectively. So, an assumption is made in this paper: the departing times of transit vehicles are uniformly distributed on timetable as exemplified in Fig. 12-13.



a) Phase difference is $0.5H_{i1}$

b) Phase difference is zero

Fig 11. A timetable with different phase difference when $H_{i2} = H_{i1}$

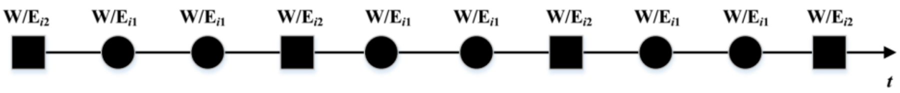


Fig 12. A timetable when $H_{i2} = m_i H_{i1}$ $m_i = 1, 2, 3, \ldots$

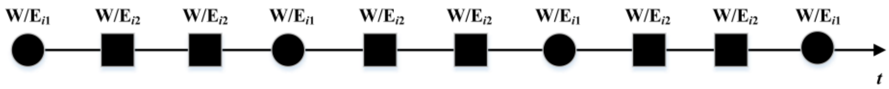


Fig 13. A timetable when $H_{i2} = m_i H_{i1}$ $m_i = 1, \frac{1}{2}, \frac{1}{3}, \ldots$

So, the line frequency from boundary i to nearer center by line W/E$_{i1}$ and line W/E$_{i2}$ are defined as f_{i1} and f_{i2}, respectively, and described by their proportion of transit vehicle departing frequency as follows (*Schmöcker et al., 2016*):

$$f_{i1} = \frac{\frac{1}{H_{i1}}}{\frac{1}{H_{i1}} + \frac{1}{H_{i2}}} \tag{18a}$$

$$f_{i2} = 1 - f_{i1} \tag{18b}$$

The integrated headway from boundary i to nearer center can be approximated by:

$$H_i = \frac{1}{\frac{1}{H_{i1}} + \frac{1}{H_{i2}}} \tag{19}$$

Appendix C Housing demand models

According to Assumption A2, this paper adopts the following Cobb-Douglas form of household's utility function (*Fallis, 1985*):

$$U(z, q) = \alpha \log(z) + (1 - \alpha) \log(q) \tag{20a}$$

subject to the household's budget constraint:

$$z + p(x)q = Y - \psi(x) \tag{20b}$$

where $U(\cdot)$ represents the household utility function; z is the quantity of household's composite good, of which the price is standardized to one; q is the quantity of housing service consumption (i.e., gross floor area); and α is a positive constant parameter reflecting the "attraction" of each factor. In the household's budget constraint (20b), x is the housing location; $p(x)$ is the average annual rental price per unit of housing at location x; Y is the household's annual income; and $\psi(x)$ is the annual travel cost per household in the two-center city, which can be expressed as:

$$\psi(x) = \mu w \big(k_I(x)\varphi_I(x) + k_{II}(x)\varphi_{II}(x)\big) \tag{20c}$$

where μ is related to household's income in the equation of $Y = 2 \times 8w\mu$, where 2 is indicates two workers per household, 8 indicates eight working hours per day; w is the annual work days, i.e., 250; $\varphi_I(x), \varphi_{II}(x)$ are the travel time per commuting trip from/to location x to/from Center I and II, respectively.

Substituting Eq. (20b) into Eq. (20a) obtains the converted households' utility model as follows:

$$U(x, q) = \alpha \log(Y - \psi(x) - p(x)q) + (1 - \alpha) \log(q) \tag{21a}$$

Solving the corresponding utility maximization problem of Eq. (21a) with respect to q and x yields the first-order optimality conditions:

$$\frac{\partial U(x,q)}{\partial q} = -\frac{\alpha p(x)}{(Y - \psi(x) - p(x)q)} + \frac{(1-\alpha)}{q} = 0 \tag{21b}$$

$$\frac{\partial U(x,q)}{\partial x} = -\frac{\alpha(\psi\prime(x) + p'(x)q)}{(Y - \psi(x) - p(x)q)} = 0 \tag{21c}$$

Rearranging Eqs. (21b-21c), we obtain:

$$q(x) = \frac{(1-\alpha)}{p(x)}\big(Y - \psi(x)\big) \tag{22a}$$

$$\psi'(x) + p'(x)q = 0 \tag{22b}$$

Substituting $q(x)$ in Eq. (22a) to Eq. (22b) obtains:

$$p'(x)\frac{(1-\alpha)}{p(x)} = -\frac{\psi\prime(x)}{(Y - \psi(x))} \tag{23}$$

Integrating both sides of Eq. (23) from 0 to x yields:

$$(1 - \alpha)\big(\log p(x) - \log\big(p(0)\big)\big) = \log\big(Y - \psi(x)\big) - \log\big(Y - \psi(0)\big) \tag{24}$$

After taking antilog, we obtain the function of housing price as follows:

$$p(x,p_0) = p_0\left(\frac{Y-\psi(x)}{Y-\psi(0)}\right)^{\frac{1}{(1-\alpha)}} \quad (25)$$

Subsequently, the housing rental price at Center II can be given by $p(D) = p_0\left(\left(\frac{Y-\psi(D)}{Y-\psi(0)}\right)\right)^{\frac{1}{(1-\alpha)}}$.

Substituting $p(x)$ in Eq. (25) to Eq. (22a) produces the function of household's housing consumption:

$$q(x,p_0) = \frac{(1-\alpha)(Y-\psi(0))^{\frac{1}{(1-\alpha)}}}{p_0}\left(Y-\psi(x)\right)^{\frac{-\alpha}{1-\alpha}} \quad (26)$$

Eqs. (25, 26) indicate that for a given p_0, the housing price ($p(\cdot)$) decreases and the housing floor space ($q(\cdot)$) per household increases as the annual travel cost ($\psi(\cdot)$) increases, and vice versa. However, in order to provide a health living environment, the minimum living area, $q_{min} = 45\ m^2$, of a family with two or more people need to be limited (*Gallent et al., 2010*).

Appendix D Housing supply models

As A3 indicates, the amount of property developers' housing supply at any location (x) is a Cobb-Douglas function of the inputs of land and capitals (*Romanos, 1977; Fallis, 1985; Li et al., 2015*):

$$\Phi(x) = \gamma F^{\beta}M^{1-\beta}, \gamma > 0, 0 < \beta < 1 \quad (27)$$

where $\Phi(x)$ is the housing supply (i.e., total floor area) at location x; F denotes the input of capital to build dwellings; M represents the input of land; and parameters γ, β are pre-specified constants. Eq. (27) can be converted into:

$$\phi(x) = \gamma e(x)^{\beta}, \gamma > 0, 0 < \beta < 1 \quad (28)$$

where $\phi(x)$ is the average floor area per unit of land area, i.e., $\phi(x) = \frac{\Phi(x)}{M}$; and $e(x)$ is the amount of capital invested on one unit of land, i.e., $e(x) = \frac{F}{M}$.

Let $r(x)$ be the value per unit of land area at location x and b is the unit cost of capital (i.e., the annual interest rate plus the annual cost of depreciation of a unit of capital), the net profit per unit of land, $\Lambda(x)$ at location x can then be given by:

$$\Lambda(x) = p(x)\phi(x) - \left(r(x) + be(x)\right) \quad (29)$$

where the price per unit of housing, $p(x)$, is obtained by Eq. (25). The first term of Eq. (29) is the total revenue from the housing rental. The final two terms are the land rent cost and the capital cost, respectively. It is observed that $\Lambda(x)$ is a strictly concave function of $e(x)$ since its second derivative with respect to $e(x)$ is negative.

According to A3, property developers aim to maximize their net profit by determining the optimal capital investment. Thus, the first-order optimality condition of the maximization problem of Eq. (29) with respect to $e(x)$ satisfies $\frac{\mathrm{d}\Lambda(x)}{de(x)} = p(x)\gamma\beta e(x)^{\beta-1} - b = 0$ in any location x, and subsequently solving the equation for $e(x)$ gives:

$$e(x,p_0) = \left(p(x)\gamma\beta b^{-1}\right)^{\frac{1}{1-\beta}} = \left(p_0\left(\frac{Y-\psi(x)}{Y-\psi(0)}\right)^{\frac{1}{(1-\alpha)}}\gamma\beta b^{-1}\right)^{\frac{1}{1-\beta}} \quad (30)$$

At the equilibrium, the quantity of the supplied housing services at each location must equal to the quantity consumed by households. Then the household residential density $n(x)$ at location x can be derived from the housing supply of $\phi(x)$ divided by the housing consumption of $q(x)$, as given by:

$$n(x,p_0) = \frac{\phi(x)}{q(x)} = \frac{\gamma e(x)^{\beta}}{q(x)} \quad (31)$$

Note that under the perfect competition, the property developers earn zero profit at any location x. Thus, derived from Eq. (29), the land value equals to property developers' revenue minus their capital investment:

$$r(x,p_0) = p(x)\gamma e(x)^{\beta} - be(x) = b^{\frac{-\beta}{1-\beta}}\left(\beta^{-1}-1\right)\left(p_0\left(\frac{Y-\psi(x)}{Y-\psi(0)}\right)^{\frac{1}{(1-\alpha)}}\gamma\beta\right)^{\frac{1}{1-\beta}} \quad (32)$$

Eqs. (30-32) define the capital investment intensity ($e(\cdot)$), residential density ($n(\cdot)$), and land value ($r(\cdot)$) at any locations as functions of the housing rental price at Center I (p_0) and the annual travel cost ($\psi(x)$). It is stated that given p_0, the capital investment intensity, residential density, and land value consistently decrease as the annual travel cost increases, and vice versa.

References

Acheampong, R. A., Silva, E. A. (2015). Land use–transport interaction modeling: A review of the literature and future research directions. *Journal of Transport and Land Use*, 8(3), 11-38.

Alonso, W. (1964). Location and Land Use: Toward a General Theory of Land Rent. Cambridge, MA: *Harvard University Press*.

Beckmann, M. J. (1974). Spatial equilibrium in the housing market. *Journal of Urban Economics*, 1(1), 99-107.

Brueckner, J. K. (1990). Growth controls and land values in an open city. *Land Economics*, 66(3), 237-248.

Cervero, R., Landis, J. (1997). Twenty years of the Bay Area Rapid Transit system: Land use and development impacts. *Transportation Research Part A: Policy and Practice*, 31(4), 309-333.

Chen, H., Gu, W., Cassidy, M., & Daganzo, C. (2015). Optimal transit service atop ring-radial and grid street networks: A continuum approximation design method and comparisons. *Transportation Research Procedia*, 7, 320-340.

Chien, S., Schonfeld, P. (1997). Optimization of grid transit system in heterogeneous urban environment. *Journal of Transportation Engineering*, 123(1), 28–35.

Chien, S., Spasovic, L.N. (2002). Optimization of grid bus transit system with elastic demand. *Journal of Advanced Transportation*, 2002, 36(1), 63-91.

Daganzo, C. F. (2010). Structure of competitive transit networks. *Transportation Research Part B: Methodological*, 44(4), 434-446.

Evans, S. P. (1973). A relationship between the gravity model for trip distribution and the transportation problem in linear programming. *Transportation Research*, 7(1), 39-61.

Fallis, G. (1985). Housing economics. *Toronto: Butterworth & Co. (Canada) Ltd*.

Fan, W., Mei, Y., & Gu, W. (2018). Optimal design of intersecting bimodal transit networks in a grid city. *Transportation Research Part B: Methodological*, 111, 203-226.

Freyss, M., Giesen, R., Munoz, J.C. (2013). Continuous approximation for skip-stop operation in rail transit. *Transportation Research Part C*, 36, 419-433.

Fujita, M. (1989). Urban economic theory: Land use and city size. *Cambridge university press*, Cambridge, UK.

Gallent, N., Madeddu, M., & Mace, A. (2010). Internal housing space standards in Italy and England. *Progress in Planning*, 74(1), 1-52.

Ghoneim, N. S., & Wirasinghe, S. C. (1987). Optimum zone configuration for planned urban commuter rail lines. *Transportation Science*, 21(2), 106-114.

Gu, W., Amini, Z., & Cassidy, M. J. (2016). Exploring alternative service schemes for busy transit corridors. Transportation Research Part B: Methodological, 93, 126-145.

Ho, H.W., Wong, S.C. (2007). Housing allocation problem in a continuum transportation system. *Transportmetrica*, 3(1), 21-39.

Holroyd, E. M. (1967). The optimum bus service: a theoretical model for a large uniform urban area. *In Proceedings of the Third International Symposium on the Theory of Traffic Flow*.

Kay, A. I., Noland, R. B., & DiPetrillo, S. (2014). Residential property valuations near transit stations with transit-oriented development. *Journal of Transport Geography*, 39, 131-140.

Kocur, G., Hendrickson, C. (1982). Design of local bus service with demand equilibration. *Transportation Science*, 16(2), 149-170.

Li, Z. C., Guo, Q. W., Lam, W. H., & Wong, S. C. (2015). Transit technology investment and selection under urban population volatility: A real option perspective. *Transportation Research Part B: Methodological*, 78, 318-340.

Li, Z. C., Lam, W. H., Wong, S. C., & Choi, K. (2012). Modeling the effects of integrated rail and property development on the design of rail line services in a linear monocentric city. *Transportation Research Part B: Methodological*, 46(6), 710-728.

Magliocca, N., McConnell, V., Walls, M., & Safirova, E. (2012). Zoning on the urban fringe: Results from a new approach to modeling land and housing markets. *Regional Science and Urban Economics*, 42(1-2), 198-210.

Medina, M., Giesen, R., & Muñoz, J. (2013). Model for the optimal location of bus stops and its application to a public transport corridor in Santiago, Chile. *Transportation Research Record: Journal of the Transportation Research Board*, (2352), 84-93.

Newell, G. F. (1979). Some issues relating to the optimal design of bus routes. *Transportation Science*, 13(1), 20-35.

Newell, G.F. (1973). Scheduling, location, transportation, and continuum mechanics: some simple approximations to optimization problems. *SIAM Journal on Applied Mathematics*, 25(3), 346-360.

Nourbakhsh, S.M., Ouyang, Y. (2012). A structured flexible transit system for low demand areas. *Transportation Research Part B: Methodological*, 46(1), 204–216.

Ouyang, Y., Nourbakhshet, S.M., & Cassidy, M.J. (2014). Continuum approximation approach to bus network design under spatially heterogeneous demand. *Transportation Research Part B: Methodological*, 68, 333-344.

Romanos, M. C. (1977). Household location in a linear multi-center metropolitan area. *Regional Science and Urban Economics*, 7(3), 233-250.

Schmöcker, J. D., Sun, W., Fonzone, A., & Liu, R. (2016). Bus bunching along a corridor served by two lines. *Transportation Research Part B: Methodological*, 93, 300-317.

Sivakumaran, K., Li, Y., Cassidy, M., & Madanat, S. (2014). Access and the choice of transit technology. *Transportation Research Part A*, 59, 204–221.

Suzuki, H., Cervero, R., & Iuchi, K. (2013). Transforming cities with transit: Transit and land-use integration for sustainable urban development. *World Bank Publications*.

Vaughan, R. (1986). Optimum polar networks for an urban bus system with a many-to-many travel demand. *Transportation Research Part B: Methodological*, 20(3), 215-224.

Voorhees, A. M. (1956). A general theory of traffic movement. *Proceedings of the Institute of Traffic Engineers*, 46-56.

Vuchic, V.R. (2005). Urban transit: Operations, planning and economics. *Hoboken, NJ: John Wiley & Sons*.

Wegener, M. (2004). Overview of land use transport models. *In Handbook of transport geography and spatial systems*. Emerald Group Publishing Limited, 127-146.

Wirasinghe, S.C., Ghoneim, N.S. (1981). Spacing of bus-stops for many to many travel demand. *Transportation Science*, 15(3), 210–221.

Yin, J., Wong, S.C., Sze, N.N., & Ho, H.W. (2013). A continuum model for housing allocation and transportation emission problems in a polycentric city. *International Journal of Sustainable Transportation*, 7(4), 275-298.

Yiu, C.Y. (2011). A spatial portfolio theory of household location choice. *Journal of Transport Geography*, 19(4), 584-590.

Yiu, C.Y., Tam, C.S. (2007). Housing price gradient with two workplaces — An empirical study in Hong Kong. *Regional Science and Urban Economics*, 37, 413-429.