A novel approach for the optimal design of skip-stop service in transit corridors

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Abstract

In this paper, we propose a discrete-continuum joint model for simultaneously optimizing the skip-stop scheme and stop locations in transit corridors. A two-line transit corridor is modeled to furnish the flexible skip-stop service, where the transit vehicles of two lines may visit any subset of the stops along the line (without the need of specifying the sequential stopping order among them as in previous literature). The discrete nature of the model comes from the discretization of the corridor into small segments, where the stops of two lines are allocated and designated by the proportions (variables to be optimized). Continuum models of system metrics are derived based on the proportion variables. The skip-stop plans of two lines are optimized to minimize the total system cost (as the sum of the transit patrons’ and agency’s cost) under heterogeneous demand. The minimization problem is efficiently solved by an integrated algorithm of the gradient descent algorithm and sequential quadratic programming algorithm. Numerical case studies demonstrate the effectiveness of the proposed model and solution algorithm in the optimal design under an artificial demand pattern. A real case study is also furnished to illustrate the huge application potential (e.g., over 20% cost saving over the current system) of the optimal design from our model.

Keywords: optimal design; skip-stop service; discrete-continuum joint model; heterogeneous demand
1. Introduction

Skip-stop service (also termed the “limited-stop service”) provides patrons faster commercial speed by operating transit lines to only stop at a subset of stops along the corridor (Larrain and Muñoz, 2016). This service scheme has long been studied in the literature (e.g. Vuchic, 1973; Afanasiev, 1983) and was also successfully implemented in many cities (Silverman, 1998; El-Geneidy and Surprenant-Legault 2010). There are several variations of skip-stop service, e.g., zonal service (Jordan and Turnquist, 1979; Furth, 1986; Larrain et al., 2015), short-turning service (Ceder, 1989; Delle Site and Filippi, 1998; Tirachini et al., 2011), and deadheading service (Cortés et al., 2011). For zonal service, a transit vehicle starts the service run by visiting all the stops within a segment, and then performs a direct travel to the city center while skipping all the stops between that segment and the city center. Short-turning service operates only in a segment of the line while skipping the remaining line entirely. In deadheading service, some transit vehicles travel in one direction of the line without taking passengers, so that they can start service earlier in the other direction. These special forms of skip-stop have limited applications for particular demand patterns. For instance, zonal service applies only to the many-to-one demand pattern; short-turning service suits for demand aggregated in partial segments; and deadheading service is designed for extremely unsymmetrical demand in two operational directions (e.g. from a city’s periphery to the city center in the morning peak hours).

More flexible skip-stop service (where a transit vehicle may visit any subset of the stops along the line) has recently been studied using sophisticated discrete models (Leiva et al., 2010; Ulusoy et al., 2010; Niu, 2011; J. Chen et al., 2015; X. Chen et al., 2015; Zhang et al., 2016; Larrain and Muñoz, 2016). Due to the voluminous input variables and constraints (e.g., the discrete OD demand and candidate stop locations), these models are usually NP-hard. Hence, they have to rely on heuristic methods, e.g. genetic algorithms and tabu search (Li et al., 1995; Lee et al., 2014), to find solutions; and the quality of those solutions (i.e. how close they are to the global optima) is often difficult to assess. Moreover, most of these studies assumed that the stop locations along the line were given, and only the transit vehicles’ routing plan (i.e. which stops to visit and which to skip) and their dispatch schedules were optimized. No discrete model, to our best knowledge, has been proposed for jointly optimizing the stop locations along the corridor and the skip-stop routing plan. A possible reason may be that the existing heuristic and metaheuristic methods are incapable of solving the complicated joint optimization problem efficiently. For sure, the joint optimization can potentially produce better (e.g. lower-cost) designs of skip-stop service.

In contrast, the continuum modeling approach for transit systems design is well known to be parsimonious, more general and computationally efficient (e.g. Daganzo, 2010; H. Chen et al., 2015). Models of this type use continuous variables or functions (e.g. density functions of stop spacing and demand) to represent the numerous parameters and variables in discrete models. Efficient solution methods can often be developed by decomposing the optimization program into many local sub-problems, which can be solved by gradient search algorithms (see e.g. Wirasinghe and Ghoneim, 1981). For the skip-stop service design problem, only a small number of continuum modeling studies are found (Gu et al., 2016). Their models
combined the optimal design of skip-stop schemes with the optimization of stop locations, and thus achieved better designs and higher cost savings as compared to the operation-level models. However, both of the above-cited works assumed uniform demand distribution over the corridor, and thus cannot furnish designs for real-world systems. Moreover, they focused on a highly special skip-stop scheme (in a sequential way of deploying stops of different lines), which limited the benefit that can be achieved for skip-stop service. This is possibly because modeling a more flexible skip-stop scheme via the continuum approach would complicate the mathematical formulation, and render the problem intractable.

In this paper, we develop a discrete-continuum joint model for simultaneously optimization of skip-stop schemes and stop locations, with respective to minimization of the generalized cost (i.e., the sum of transit patrons’ and agency’s cost). The model takes the advantages of generality and parsimony from the continuum models. However, if formulated fully in the continuum way, the problem would be very difficult to solve since it cannot be decomposed into local subproblems. Hence, we discretize the continuum-fashion program and convert it to a nonlinear program, which can be solved efficiently using commercial solvers. The model accepts realistic, spatially heterogeneous demand as inputs, and accounts for a very flexible layout of skip-stop lines. We show through numerical case studies that our model can be solved more efficiently than conventional discrete models with numerous binary variables. A real case study demonstrates that implementation-ready designs of skip-stop service can be furnished by our work.

2 Methodology

Without loss of generality, consider a linear transit corridor of length \( L \), which has two skip-stop routes\(^1\): route A and route B, and three types of stops: A-type stops, B-type stops, and transfer (T-type) stops, as exemplified in Figure 1. Vehicles of route A only visit A- and T-type stops, and vehicles of route B visit B- and T-type stops. Apparently, a patron who travels from an A-type stop to a B-type stop will have to make a transfer at a T-type stop to complete her trip. Note in Figure 1 that the three types of stops can appear in an arbitrary order along the corridor.

![Figure 1. Skip-stop service with two routes in a transit corridor](image)

To facilitate the modeling, we adopt the following assumptions that were commonly used in the literature (Freyss et al., 2013; Gu, et al., 2016): i) a patron always accesses or egresses the transit system at the nearest stop to her origin or destination, regardless of the stop type; ii) a

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\(^1\) Corridors with more than two skip-stop routes can be modeled in a similar way. However, in practice these services are difficult to be implemented due to its complexity (for transfer passengers) and high agency cost.
transit vehicle spends a constant time, $\tau$ (h), at each stop for loading and unloading patrons, including the time lost to the vehicle’s acceleration and deceleration\(^2\); iii) the patrons arrive uniformly to their origin stops and always take the first vehicle they meet; iv) a patron only makes the transfer (at the first transfer stop encountered) if there is no direct service from her origin to her destination; and v) vehicles of the two skip-stop routes are not coordinated at the transfer stops.

The demand is represented by the conventional origin-destination (OD) matrix $(m \times m)$ that can be obtained from a transit demand survey. This implies that the corridor is divided into $m$ consecutive segments with distinct length $l_i$, indexed from left to right by $i \in \{1, 2, \ldots, m\}$ ($\sum_{i}^{m} l_i = L$). We denote $\lambda_{i,j}$ as the total demand (patrons/h) from segment $i$ to $j$. The demand is assumed to be uniformly distributed within each segment\(^3\). The decision variables of the skip-stop service design problem include: the headways for route A and route B, $H_A$ and $H_B$, respectively; the number of stops in segment $i$, $n_i$; proportions of A-, B-, and T-type stops in segment $i$, $a_i, b_i, t_i$, respectively. We seek to minimize the generalized cost of the transit corridor, which consists of the patrons’ trip cost and the agency cost. Detailed model formulations are given below.

### 2.1 Patron cost metrics

The patrons’ cost includes four components (Daganzo, 2010): i) the access/egress cost, $UC_A$; ii) the waiting cost at the origin and transfer stops, $UC_W$; iii) the in-vehicle travel cost, $UC_I$; and iv) the transfer penalty, $UC_T$.

#### i) Access and egress cost

Consider segment $k$ in the corridor, the access and egress cost equals to the total walking time per hour in the segment. We donate patrons’ average walking speed as $v_w$, the number of stops within the segment as $n_k$, and the length of the segment as $l_k$. Thus, the average walking time for either a patron’s access or egress is $\frac{l_k}{4n_kv_w}$. The access and egress cost in the segment is then given by $\sum_{j=1}^{m} \lambda_{k,j} \cdot \frac{l_k}{4n_kv_w} + \sum_{i=1}^{m} \lambda_{i,k} \cdot \frac{l_k}{4n_kv_w}$.

Thus, the total access cost and egress cost of the corridor is

$$UC_A = \sum_{j=1}^{m} \left[ \frac{l_j}{4vwn_i} \left( \sum_{j=1}^{m} (\lambda_{i,j} + \lambda_{j,i}) \right) \right]$$  \hspace{1cm} (1)

#### ii) Waiting cost at the origin and transfer stops

The waiting cost is the expected waiting time for all patrons per hour, including the waiting

\(^2\) In some papers (Cipriani et al., 2012; Meng and Qu, 2013), a bus’s dwell time at a stop was assumed as a linear function of the number of boarding patrons at the stop. With modest changes, the methodology to be presented in this paper can still be applied if this alternative assumption is used instead.

\(^3\) A demand concentration point (e.g. the entrance to a residential community or a large shopping mall) can be represented by a very small-sized segment.
time at origin stops and transfer waiting time at intermediate transfer stops. Since we assume the arrivals of patrons are uniformly distributed, the waiting time for a patron is considered as half of the headway of the target route.

Specifically, for an individual patron, her waiting cost depends on her trip pattern, which belongs to 9 types of trip patterns in total, according to the possible combinations of the three types (i.e., A-, B-, and T-type) of her origin and destination stops. Given the proportion of the A-, B- and T-type stop at segment \( i \) being \( a_i, b_i \) and \( t_i \), the probability of each type of trips and the corresponding expected wait time can be easily estimated. For example, consider a trip from origin segment \( i \) to destination segment \( j \), the probability of boarding at A-type stop and alighting at A-type stop is \( a_i a_j \). The expected wait time is thus the product of the probability and the waiting time for this trip, equaling \( a_i a_j \frac{H_A}{2} \). For another trip from A-type stop to B-type stop that involves a transfer, the probability is \( a_i b_j \) and the expected waiting time should include the waiting time at the transfer stop, i.e., \( a_i b_j \frac{H_A + H_B}{2} \). Table 1 summarizes the probabilities and the expected waiting time for all types of trip patterns. Note that it can be easily verified that their probabilities add up to one.

Table 1. The probability of 9 types of trip patterns and the waiting time of each trip

<table>
<thead>
<tr>
<th>Trip type</th>
<th>Origin stop type</th>
<th>Destination stop type</th>
<th>Probability</th>
<th>Waiting time</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>A</td>
<td>A</td>
<td>( a_i a_j )</td>
<td>( H_A/2 )</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>B</td>
<td>( a_i b_j )</td>
<td>( (H_A + H_B)/2 )</td>
</tr>
<tr>
<td>3</td>
<td>A</td>
<td>T</td>
<td>( a_i t_j )</td>
<td>( H_A/2 )</td>
</tr>
<tr>
<td>4</td>
<td>B</td>
<td>A</td>
<td>( b_i a_j )</td>
<td>( (H_A + H_B)/2 )</td>
</tr>
<tr>
<td>5</td>
<td>B</td>
<td>B</td>
<td>( b_i b_j )</td>
<td>( H_B/2 )</td>
</tr>
<tr>
<td>6</td>
<td>B</td>
<td>T</td>
<td>( b_i t_j )</td>
<td>( H_B/2 )</td>
</tr>
<tr>
<td>7</td>
<td>T</td>
<td>A</td>
<td>( t_i a_j )</td>
<td>( H_A/2 )</td>
</tr>
<tr>
<td>8</td>
<td>T</td>
<td>B</td>
<td>( t_i b_j )</td>
<td>( H_B/2 )</td>
</tr>
<tr>
<td>9</td>
<td>T</td>
<td>T</td>
<td>( t_i t_j )</td>
<td>( \frac{H_A H_B}{H_A + H_B} )</td>
</tr>
</tbody>
</table>

Therefore, the total waiting cost can be expressed by:

\[
U \mathcal{C}_W = \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{n} \left( H_A \left( a_i + b_i a_j + t_i a_j \right) + H_B \left( b_i + a_i b_j + t_i b_j \right) + \frac{H_A H_B}{H_A + H_B} t_i t_j \right) \lambda_{i,j} \tag{2}
\]

### iii) In-vehicle travel cost

The in-vehicle travel cost is the total travel time for patrons spent in the vehicles per hour, which is also influenced by the type of trip patterns. For instance, some short-distance trips (that have trip ends at A- and B-type stops but do not have a transfer stop in the trip range) may have to backtrack to/from a nearby transfer stop to accomplish their trips. Extra costs are caused. All other types of trips, involving a transfer or not, are direct trips without need to travel reversely. We next derive the cost metrics for the direct trips and backtracking trips, respectively.

First, for the direct trips, let \( \bar{\sigma}_k \) be the average on-board flow in segment \( k \), which is
composed of $\delta_k^A$ in route A and $\delta_k^B$ in route B. We now derive $\delta_k^A$ for route A under two cases, i.e., trips without and with a transfer, as follows ($\delta_k^B$ can be determined in a similar way). Consider a trip from segment $i$ to segment $j$, the probability of the patron taking route A without a transfer is $p_{i,j}^{A1} = a_i a_j + a_i t_j + t_i a_j + t_i t_j \frac{H_B}{H_A + H_B}$, where $\frac{H_B}{H_A + H_B}$ yields the probability of a patron boarding on route A for a T-type to T-type trip; other items in $p_{A1}$ are straightforward. If the trip involves a transfer (e.g., from A-type stop to B-type stop or vice versa), the on-board flow will be the sum of that in the two routes. Since we assumed that patrons always transfer at the first encountered transfer stop, the on-board flow of route A with a transfer will be mainly dictated by the trips from B-type stop to A-type stop. Thus, the probability of a trip taking route A with a transfer can be approximated by $p_{i,j}^{A2} = b_i a_j$ (where $i, j$ denote the origin and destination segments, respectively). Thus, the total on-board flow of route A at the right boundary of segment $k$, $\sigma_k^A$, can be represented by:

$$
\sigma_k^A = \sum_{i=1}^{m-k} \sum_{j=1}^{m-k} [p_{i,j}^{A1} + p_{i,j}^{A2}] \lambda_{i,j} + \sum_{i=1}^{m-k} \sum_{j=1}^{m-k} [p_{i,j}^{A1} + p_{i,j}^{A2}] \lambda_{i,j}, \quad k = 1, 2, \ldots, m - 1;
$$

$$
\sigma_k^A = 0, \quad k = 0 \text{ or } m.
$$

To simplify the notations, we introduce a series of supplementary OD matrices $U_k$, given by:

$$
U_k = \begin{bmatrix}
0 & 0 & 0 & \lambda_{1,k+1} & \cdots & \lambda_{1,m} \\
0 & \cdots & 0 & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \lambda_{k+1,k+1} & \cdots & \lambda_{k+1,m} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \lambda_{m+1,k} & \cdots & \lambda_{m,k} & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}, \quad k = 1, 2, \ldots, m - 1;
$$

$$
U_k = 0, \quad k = 0 \text{ or } m.
$$

Thus, $\sigma_k^A$ can be reformulate as $\sigma_k^A = \sum_{i=1}^{m-k} \sum_{j=1}^{m-k} [p_{i,j}^{A1} + p_{i,j}^{A2}] \mu_{i,j}^k$, where $\mu_{i,j}^k$ is the entry $i,j$ of the $k$th supplementary OD matrix $U_k$. We then can calculate the average on-board flow in segment $k$, $\sigma_k^A$, by the mean value of flow on the left and right boundaries of the segment:

$$
\bar{\sigma}_k^A = (\sigma_k^A + \sigma_{k-1}^A) / 2 = \sum_{i=1}^{m-k} \sum_{j=1}^{m-k} [p_{i,j}^{A1} + p_{i,j}^{A2}] \bar{\mu}_{i,j}^k, \quad k = 1, 2, \ldots, m
$$

where we define $\bar{\mu}_{i,j}^k = (u_{i,j}^k + u_{i,j}^{k-1}) / 2, k = 1, 2, \ldots, m$.

Similarly, the on-board flow for route B can be written as:

$$
\sigma_k^B = \sum_{i=1}^{m} \sum_{j=1}^{m} (p_{i,j}^{B1} + p_{i,j}^{B2}) \bar{\mu}_{i,j}^k, \quad k = 1, 2, \ldots, m
$$

where $p_{i,j}^{B1} = b_i b_j + b_i t_j + t_i b_j + t_i t_j \frac{H_A}{H_A + H_B}$; $p_{i,j}^{B2} = a_i b_j$.

Therefore, the cost of the direct trips in segment $k$ is expressed by:

$$
\Gamma_k = \left[ \sigma_k^A \left( \frac{k}{v} + \tau (a_k + t_k) n_k \right) + \sigma_k^B \left( \frac{k}{v} + \tau (b_k + t_k) n_k \right) \right]
$$

where $v$ is the average cruising speed of a vehicle (which is assumed to be indifferent for
route A and B); and \((a_k + t_k)n_k\) and \((b_k + t_k)n_k\) yield the total number of stops in segment \(k\) for route A and B, respectively.

For the backtracking trips, consider segment \(k\) in the corridor, the number of transfer stops is \(t_kn_k\) and the average transfer stop spacing can be approximated as \(s_k = \min\left(\frac{L}{x_k} + \frac{I_k}{2t_kn_k} \right) - \max\left(0, x_k - \frac{I_k}{2t_kn_k} \right)\), where \(x_k\) is the coordinate of the center of segment \(k\). Note that \(s_k\) may span more than one segment if \(t_kn_k\) is small. We index the left-side segment of \(s_k\) spanned as \(bl_k = I \left( \max \left(0, x_k - \frac{I_k}{2t_kn_k}\right) \right)\) and right-side segment as \(br_k = I \left( \min \left(\frac{L}{x_k} + \frac{I_k}{2t_kn_k}\right) \right)\), where \(I(x)\) is an index function that maps to the index of the segment that contains coordinate \(x\).

Thus, the number of backtracking trips within \(s_k\) is \(\sum_{i=bl_k}^{br_k} \sum_{j=bl_k}^{br_k} (a_i b_j + b_i a_j) \lambda_{i,j}\). Again, we reformulate the notations using a series of complementary OD matrices \(R_k\), given by:

\[
R_k = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & \lambda_{bl_k,bl_k} & \cdots & \lambda_{bl_k,br_k} & 0 \\
0 & \cdots & \cdots & \cdots & 0 \\
0 & \lambda_{br_k,bl_k} & \cdots & \lambda_{br_k,br_k} & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Thus, the expression can be written as \(\sum_{i=1}^{m} \sum_{j=1}^{m} (a_i b_j + b_i a_j) r_{i,j}^k\), where \(r_{i,j}^k\) is the entry \(i,j\) of the \(k\) th supplementary OD matrix \(R_k\). Then, we calculate the average backtracking density with respective to segment \(k\), \(\bar{x}_k\) (trips per unit distance), given by:

\[
\bar{x}_k = \frac{\sum_{i=1}^{m} \sum_{j=1}^{m} (a_i b_j + b_i a_j) r_{i,j}^k}{\sum_{i=bl_k}^{br_k} \lambda_{i,j}} \tag{6}
\]

Since the distance interval, namely a bay, of two consecutive transfer stops should be designed small (Freyss et al., 2013), we approximate the trips distributed uniformly in the bay for simplifying the derivation of the backtracking cost. Thus, the total backtracking vehicle distance is \(\frac{1}{3} s_k \bar{x}_k\), where \(\frac{1}{3} s_k\) is the average backtracking distance in segment \(k\) (see detailed derivation in Gu et al., 2016). Acknowledging that half of backtracking flow is on route A and half on route B, the average probability of stopping for a backtracking trip is approximately 0.5. Therefore, the backtracking cost in segment \(k\) can be given by \(\varphi_k = \left(\frac{tk}{v} + \frac{1}{2} n_k\right) \frac{1}{3} s_k \bar{x}_k\). Consideration shows that \(\varphi_k\) can be further simplified using a tight upper bound of \(s_k\), \(\frac{a_tk}{tkn_k + \frac{1}{2}}\) where \(\alpha = \max_k \frac{tk+1}{tk}\) (see Appendix for details), and re-written as:

\[
\varphi_k = \frac{a \frac{l_k}{v}}{tkn_k + \frac{1}{2}} \frac{l_k}{tkn_k + \frac{1}{2}} \bar{x}_k \tag{7}
\]

This simplification gives \(\varphi_k\) the property of being differentiable, which facilitate the problem solving.
Combing the above cost metrics, the total in-vehicle travel cost is given by:

\[ U_C = \sum_{k=1}^{m} [\Gamma_k + \varphi_k] \]  

(8)

where the direct and backtracking trip cost, \( \Gamma_k \) and \( \varphi_k \), are given by equations (5,7).

iv) Transfer penalty

We also consider an extra cost penalized for transfers. It’s because people always feel inconvenient for transfers and some transfers may also need extra walking or buying an extra ticket.

The probability of a trip that involves a transfer is \( p_{ij} = a_i b_j + b_i a_j \), such that the transfer penalty is expressed by:

\[ U_C = \sum_{i=1}^{m} \sum_{j=1}^{m} C_t p_{ij} \lambda_{i,j} \]  

(9)

where the \( C_t \) is the penalty coefficient (h/transfer).

2.2 The agency costs

The agency cost consists of: i) the distance-based vehicle operating cost (e.g. fuel cost), \( AC_K \); ii) the time-based vehicle operating cost (e.g. amortized vehicle purchase cost and staff wages), \( AC_H \); iii) the amortized line infrastructure cost (e.g. busway or rail tracks), \( AC_i \); and iv) the amortized stop infrastructure cost, \( AC_S \). They are formulated as follows:

\[ AC_K = \frac{2\pi_{vl}}{\mu} \left( \frac{1}{H_A} + \frac{1}{H_B} \right) \]  

(10)

\[ AC_H = \frac{2\pi_M}{\mu H_A} \sum_{i=1}^{m} \left( \frac{1}{v} + \tau (a_i + t_i) n_i \right) + \frac{2\pi_M}{\mu H_B} \sum_{i=1}^{m} \left( \frac{1}{v} + \tau (b_i + t_i) n_i \right) \]  

(11)

\[ AC_i = \frac{2\pi_{h}}{\mu} \]  

(12)

\[ AC_S = \frac{\pi_S}{\mu} \sum_{i=1}^{m} n_i \]  

(13)

where \( \mu \) is the value of time ($/h); \( \pi_v \) and \( \pi_M \) are the unit operating costs per vehicle-km and per vehicle-hour of service for each hour of service, respectively; \( \pi_t \) and \( \pi_s \) are the unit infrastructure construction and maintenance costs per km of line infrastructure and per stop, respectively, amortized for each hour of service.

2.3 Problem formulation

The generalized cost optimization problem is formulated as follows:

\[ \min SC = UC_A + UC_W + UC_I + UC_T + AC_K + AC_H + AC_I + AC_S \]  

(14a)

subject to:

\[ H_A \geq H_B \geq H_{\text{min}} \]  

(14b)

\[ \frac{K}{H_A} \geq d_i^d + \phi_i, \forall i \in 1, 2, \ldots, m \]  

(14c)

\[ \frac{K}{H_B} \geq d_i^b + \phi_i, \forall i \in 1, 2, \ldots, m \]  

(14d)
where $H_{min}$ is the minimum service headway for each skip-stop route; $K$ is the vehicle’s passenger-carrying capacity; $\phi_i$ is the maximum backtracking flow in segment $i$, which is approximately equal to half of the backtracking flow in the segment, $\phi_i = \frac{a_i}{2E_kn_k+\lambda_k}$ (assuming demand is uniformly distribution within the segment; see Gu et al. (2016) for more discussion). Note that to break the symmetry of route A and route B, we restrict $H_A \geq H_B$ in constraint (14b). Constraints (14c-d) ensure the number of passengers onboard a transit vehicle never exceeds the vehicle capacity. Constraints (14e) specify that the proportions of all the three types of stops in each segment sum up to 1.

3 Solution method

Noticing in (14) that the cost metrics of backtracking trips, $\phi_i$ and $\varphi_i$ $(i \in \{1, 2, \ldots, m\})$, cannot be expressed as analytical functions of the decision variables $x = (a_1, \ldots, a_m, b_1, \ldots, b_m, t_1, \ldots, t_m, n_1, \ldots, n_m)^T$, we will employ iteration method to find the solution assuming in each iteration $\phi_i$ and $\varphi_i$ as given values and updating them in next. Now first consider that $H_A$ and $H_B$ are fixed. The remaining problem becomes a non-convex nonlinear program with linear constraints in the form of:

\[
\begin{align}
\min f(x) & \quad \text{subject to } Gx \leq g \text{ and } Ex = e \\
\end{align}
\]

where $G$, $E$ are matrices and $g$, $e$ are vectors with respect to constraints (14e-f). A local optimum solution of $x^*$ can be quickly found by a standard solver, e.g. CPLEX. The backtracking flow and time $\phi_i$ and $\varphi_i$ $(i \in \{1, 2, \ldots, m\})$ are then updated using the obtained the optimal $x^*$. The process is iterated until the solution converges. This iteration process is then embedded in an outermost problem where the headway variables $H_A$ and $H_B$ are optimized. The detailed steps are summarized below,

1. Randomly initialize $H_A, H_B, x$;
2. Calculate $\phi_i$ and $\varphi_i$ by the given $H_A, H_B, x$;
3. Resolve the nonlinear program (14a-b) by sequential quadratic programming (SQP) method, and find the optimal $x^*$;
4. Update $\phi_i$ and $\varphi_i$, and go back to step 3 until convergence;
5. Calculate the gradient of the total cost (14a) with respective to $H_A, H_B$, and update their values through gradient descent algorithm, then go back to step 2 to continue the algorithm. If the gradient descent procedure converges or the constraints (14b-d) are violated, the algorithm ends.

Note that the solution obtained from the above algorithm may not be an implementation-ready design, because the number of A-type stops in segment $i$, $a_in_i$, may not be an integer, for instance. We therefore develop the following recipe to convert the optimal solution to a realistic design of skip-stop system.
We first determine the total number of all stops in each segment. Let \( \tilde{n}_k \) be the rounded number of stops, it can be obtained by \( \tilde{n}_k = \lfloor \sum_{p=1}^{k} n_p - \sum_{p=1}^{k-1} n_p \rfloor \), where \( \lfloor \cdot \rfloor \) takes the floor of the fraction. We then allocate specific stop types in each segment \( k \) by matching the corresponding proportion values (i.e., \( a_k, b_k, t_k \)). Let \( Y = \{A, B, T\} \) be the set of stop types, and \( O_s \) be the set of assigned stop type for stop \( s \) (\( s = 1, 2, ..., \sum_{k=1}^{m} \tilde{n}_k \)). We introduce a series of complementary functions to represent the probability of a stop being a particular stop type, i.e. \( d^k_A = d^k_B = d^k_T \), where \( i_k \) is the index of the first stop in segment \( k \), and \( j_k \) is the index of the last stop in segment \( k \). In addition, we specify \( g_s(y), y \in Y \) to represent the residual probability for stop type \( y \) at stop \( s \). Detailed procedure of the stop allocation is summarized as follows:

**Step 1.** Initialize \( s = 1 \); set residual probabilities for the first stop. \( g_1(y) = d_1(y), \forall y \in Y \);

**Step 2.** Find the stop type \( y^* \) that has the highest residual probability, \( y^* = \arg \max g_s(y) \); subtract the residual probability \( g_s(y^*) \) from 1, \( g_s(y^*) = g_s(y^*) - 1 \) and assign \( y^* \) to \( O_s \).

**Step 3.** Let \( s = s + 1 \) and update residual probability. \( g_s(y) = g_{s-1}(y) + d_s(y), \forall y \in Y \). Stop if \( s \) is the last stop of the corridor; otherwise go to step 2.

### 4. Numerical case studies

#### 4.1 Chessboard demand pattern

We first study an idealized demand pattern for sanity check of our models and algorithms. The demand is given by an OD matrix with a chessboard shape, as shown in Figure 2, in a rail corridor of length 20 km. The OD matrix consists of six groups of OD pairs with low demand density (yellow squares) and two groups with high demand density (black squares). The demand heterogeneity is further represented by specifying that no patron travels from low demand groups to high demand groups and vice versa.

![Figure 2. Idealized chessboard demand pattern](image)
Intuitively, the optimal design of this special demand pattern should have two independent routes: one serves only the low demand groups and the other the high demand groups. This intuition is well reflected in the optimal results: Figure 3 shows the optimized stop spacings (also locations) under the demand pattern, in which the blue circles represent the stop locations of route A, the red squares stop locations of route B. No transfer stops are needed. The validity of our model is demonstrated in the specially designed scenario.

![Figure 3. Optimal stop spacing for the idealized chessboard demand pattern](image)

We then compare the optimal design of our skip-stop model with that of the traditional all-stop design model. The results are listed in Table 2. Note first that the optimal skip-stop design has more stations and higher vehicle frequency than the optimal all-stop design. The in-vehicle travel cost $UC_I$, however, shows a great percentage of cost saving in skip-stop design (17.1%). It is mainly because patrons experience fewer stops in the optimized skip-stop service, and consequently spend less in-vehicle travel time than those in the all-stop design. In addition, skip-stop design also saves patrons’ walking time, i.e. $UC_A$, thanks to the shorter stop spacings. Comparisons between other cost components also show some negative benefit of the skip-stop service in both patrons’ and agency’ cost, but overall the positive benefits overwhelm and generate 8.3% saving in the total system cost. Even more considerable saving (16.1%) is witnessed from the perspective of patrons.

<table>
<thead>
<tr>
<th>Table 2 Comparison of skip-stop design and all-stop design</th>
</tr>
</thead>
<tbody>
<tr>
<td>Skip-stop design</td>
</tr>
<tr>
<td>$H_A$ (min)</td>
</tr>
<tr>
<td>$H_B$ (min)</td>
</tr>
<tr>
<td>Number of stations</td>
</tr>
<tr>
<td>$UC_A$ (h)</td>
</tr>
<tr>
<td>$UC_W$ (h)</td>
</tr>
<tr>
<td>$UC_I$ (h)</td>
</tr>
<tr>
<td>$UC_T$ (h)</td>
</tr>
<tr>
<td>$AC_K$ (h)</td>
</tr>
<tr>
<td>$AC_H$ (h)</td>
</tr>
<tr>
<td>$AC_I$ (h)</td>
</tr>
<tr>
<td>$AC_S$ (h)</td>
</tr>
<tr>
<td>User cost (h)</td>
</tr>
<tr>
<td>Agency cost (h)</td>
</tr>
<tr>
<td>Generalized cost (h)</td>
</tr>
</tbody>
</table>
4.2 Realistic demand pattern

We applied our model to a realistic case of Metro Line 11 in Shenzhen, China, as shown in Figure 4. The line is of 51 km long and contains 18 stations in total. The existing spacings between all the stations along the line is shown in Figure 5, in which the blue circles represent the stop locations and the positive direction of $x$ axis means the westbound direction ($x = 0$ is the coordinate of station 1). The real station-to-station OD demand is collected during peak periods in Dec. 1, 2016, as visualized in Figure 6. The corresponding numbers of boarding and alighting passengers at each station are depicted in Figure 7. It is noticed for station 3, 6 and 10 that both the boarding and alighting volumes are significantly smaller than that at other stations.

![Figure 4. Metro Line 11, Shenzhen, China](image)

![Figure 5. Station spacings of Metro Line 11](image)
Figure 6. OD matrix of Metro Line 11

Figure 7. Boarding and alighting passengers of Metro Line 11

Under the given demand pattern, the optimized results of our skip-stop model is shown in Figure 8, where the blue circles are the station locations of route A, the red squares denote the stations of the route B, and the black circles are the transfer stations. The optimal station spacings are in the range of 0.7 km and 5 km and vary along the corridor. It is observed that T-type stations are relatively densely deployed at the two ends of the corridor. This is because at corridor edges the high boarding and alighting demand (relative to the low on-board passing demand) call for easy access and egress (other than fast commercial speed), to which T-type station best serve. Another observation is that the optimal station spacings in segments of 5-10 km, 15-20 km and 32-37 km are much larger than that in other parts. This result echoes the significant low demand profile at station 3, 6, 10 in the existing line.
The performance of our optimal design is compared with that of the existing system. To be fair, the headway of the existing metro line is optimized. The comparison further includes an all-stop scenario under the optimal design (with respect to stop spacings and headways). The results are listed in Table 4. It is observed that our model brings about considerable benefits (over 20% and 3%) to the existing system as well as to the all-stop scenario by operating a two-line skip-stop service. Not surprisingly, the newly skip-stop service requires the highest agency cost due to the increased dispatching frequency and number of stations. The overwhelming cost savings (15% and 11%) come from the reduced access and egress cost $UC_A$ and the in-vehicle travel cost $UC_I$. The results imply huge application potential of the proposed model.

### Table 4 Comparison between skip-stop design and all-stop design

<table>
<thead>
<tr>
<th></th>
<th>Existing line</th>
<th>All-stop design</th>
<th>Skip-stop design</th>
<th>Cost saving (skip-stop vs. existing)</th>
<th>Cost saving (skip-stop vs. all-stop)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_A$ (min)</td>
<td>0.1</td>
<td>0.07</td>
<td>0.09</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$H_B$ (min)</td>
<td>-</td>
<td>-</td>
<td>0.09</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Number of stations</td>
<td>18</td>
<td>44</td>
<td>52</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$UC_A$ (h)</td>
<td>17521</td>
<td>6419</td>
<td>5423</td>
<td>69.0%</td>
<td>15.5%</td>
</tr>
<tr>
<td>$UC_W$ (h)</td>
<td>587</td>
<td>788</td>
<td>1021</td>
<td>-73.9%</td>
<td>-29.6%</td>
</tr>
<tr>
<td>$UC_I$ (h)</td>
<td>8474</td>
<td>10647</td>
<td>9471</td>
<td>-11.8%</td>
<td>11.0%</td>
</tr>
<tr>
<td>$UC_T$ (h)</td>
<td>0</td>
<td>0</td>
<td>122</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$AC_K$ (h)</td>
<td>222</td>
<td>317</td>
<td>493</td>
<td>-122.1%</td>
<td>-55.5%</td>
</tr>
<tr>
<td>$AC_H$ (h)</td>
<td>322</td>
<td>594</td>
<td>824</td>
<td>-155.9%</td>
<td>-38.7%</td>
</tr>
<tr>
<td>$AC_I$ (h)</td>
<td>7983</td>
<td>7983</td>
<td>7983</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>$AC_S$ (h)</td>
<td>701</td>
<td>1686</td>
<td>1999</td>
<td>-185.2%</td>
<td>-18.6%</td>
</tr>
<tr>
<td>User cost (h)</td>
<td>26582</td>
<td>17854</td>
<td>16037</td>
<td>39.7%</td>
<td>10.2%</td>
</tr>
<tr>
<td>Agency cost (h)</td>
<td>9283</td>
<td>10580</td>
<td>11299</td>
<td>-21.7%</td>
<td>-6.8%</td>
</tr>
<tr>
<td>Generalized cost (h)</td>
<td>35810</td>
<td>28434</td>
<td>27336</td>
<td>23.7%</td>
<td>3.9%</td>
</tr>
</tbody>
</table>

5. Conclusions

We extended the previous continuum model of skip-stop service design (Mei et al., 2018) to a discrete-continuum joint model, in which continuum models are built using continuous decision
variables, e.g., stop proportions in each discretized segments of the corridor. Thus, the model retains the solution efficiency and gains the capability of producing implementation-ready designs. The proposed model allows more flexibility of design, in which the stopping order of transit vehicles from multiple lines may vary over the corridor, other than being specified as in previous literature (Mei et al., 2018; Gu et al., 2016; Freyss, 2013). Other heterogeneous designs remain, e.g., location varying stop spacings for non-transfer and transfer stops.

To resolve the optimization problem, we first discretize the problem into a quadratic program, and then find the solution by an integrated algorithm of the gradient descent algorithm and sequential quadratic programming algorithm. During the discretization course, the optimal design is also transformed into specific stop locations of different lines. The effectiveness and the applicability of our model are demonstrated in numerical case studies and a real case study in Shenzhen City (China).

Potential applications of our model are seen such as for optimizing the skip-stop routing plans for existing transit systems, where stop locations are fixed; and for time-dependent operations during different times of day. Extensions can also be made to consider different transit modes in cooperation of a skip-stop service; and the schedule coordination in transfer stops.

Acknowledgments

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Appendix

Here, we prove that $\frac{\alpha l_k}{t_k n_k + L}$ is a tight upper bound of $s_k = \min \left( L, x_k + \frac{l_k}{2t_k n_k} \right) - \max \left( 0, x_k - \frac{l_k}{2t_k n_k} \right)$, where $\alpha = \max \frac{l_k + 1}{l_k}$.

1) If $\frac{l_k}{2t_k n_k} \geq L$, then $s_k = L - 0 = L$. Since $\frac{l_k}{2t_k n_k} \geq L \Rightarrow l_k \geq 2t_k n_k L$, we have $\frac{\alpha l_k}{t_k n_k + L} \geq \frac{2\alpha l_k}{l_k + 2} L \geq L$.

2) If $\frac{L}{2} \leq \frac{l_k}{2t_k n_k} \leq L$, we have the following cases:

(a) If $x_k + \frac{l_k}{2t_k n_k} \leq L$ and $x_k - \frac{l_k}{2t_k n_k} \leq 0$, then $x_k \leq L - \frac{l_k}{2t_k n_k}$, $s_k = x_k + \frac{l_k}{2t_k n_k}$.

Since $x_k + \frac{l_k}{2t_k n_k} \leq L$, we have $s_k \leq L$. Because $\frac{L}{2} \leq \frac{l_k}{2t_k n_k} \leq L$, the upper bound

$\frac{\alpha l_k}{t_k n_k + L} \geq \frac{\alpha l_k}{l_k + 2} L \geq L \geq s_k$.

(b) If $x_k + \frac{l_k}{2t_k n_k} \leq L$ and $x_k - \frac{l_k}{2t_k n_k} \geq 0$, then $x_k = 0$. 

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(c) If $x_k + \frac{i_k}{2t_k n_k} \geq L$ and $x_k - \frac{i_k}{2t_k n_k} \leq 0$, then $L - \frac{i_k}{2t_k n_k} \leq x_k \leq \frac{i_k}{2t_k n_k}, s_k = L$

Also, such as in (a) the upper bound $\frac{\alpha \cdot L}{t_k n_k + L} \geq L = s_k$.

(d) If $x_k + \frac{i_k}{2t_k n_k} \geq L$ and $x_k - \frac{i_k}{2t_k n_k} \geq 0$, then $x_k = \emptyset$.

3) If $\frac{i_k}{2t_k n_k} \leq \frac{t}{2}$, then let $x_k = L - x_k$ and substitute $x_k$ in $s_k$. Then the derivation is just as similar as 2).

References


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