A hybrid stochastic approach for train trajectory reconstruction

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Abstract The next generation of railway systems will require more and more accurate information on train operation. These requirements are essential for the introduction of automatic processes into traffic management and train operation, the optimal use of infrastructure capacity and energy, and overall the introduction of data-driven approaches into rail operation. Train trajectories are a key information for operation, and their collection constitutes a primary source of information for offline procedures such as timetables generation, driving behaviour analysis and models’ calibration. Unfortunately, current train trajectory data are often affected by measurement errors, missing data and, in some cases, incongruence between dependent variables. To overcome this problem, a trajectory reconstruction problem must be solved, before using trajectories for any further purpose.

In the present paper, a new hybrid stochastic trajectory reconstruction is proposed. On-board monitoring data on train position and velocity (kinematics) are combined with data on power used for traction and feasible acceleration values (dynamics). A fusion of those two types of information is performed by considering the stochastic characteristics of the data, via smoothing techniques. A promising potential use is seen especially in those cases where information on continuous train positions is not available or unreliable (e.g. tunnels, canyons, etc.).

Keywords: Rail operation, Train trajectories, reconstruction, maximum a posteriori
1 Introduction and literature review

Railway industry is taking definite steps towards digitalization, to keep being competitive in the future mobility market. The advances in timetable generation and rail traffic management will enhance the system performances by providing better infrastructure use, more adherence to scheduled operation, more potential for recovering delays. A fundamental prerequisite for the implementation of these tools is the collection of reliable and precise data on train operation. Unfortunately, these data are typically affect by noises and errors, and an a posteriori data processing is needed before their use. The present paper proposes a novel hybrid stochastic approach that aims at reconstructing train operation information, specifically train trajectories, from data collected onboard. Main purpose is to have congruent, reliable, likely ground truth to help developing more performing timetables and traffic management systems (TMSs). The approach proposed in this paper is specifically addressed to support offline elaborations.

Train trajectories are the main information needed for traffic planning and management. From these data it is possible to accurately plan train operation according to real system performances, and thus to reduce the use of online procedures for generating rescheduling solutions. Rescheduling procedures are used at traffic management level to mitigate the impacts (e.g. increasing delays) of possible deviations from timetable on the rail traffic [17]. Recently, many promising results of multi-objective optimization models have been shown, i.e. models able to both solve conflicts and reduce delays, and at the same time to enhance system performances such as capacity use [18] or energy consumption [19, 20]. Regarding this latter aspect, more detailed data on train trajectory are required; this means that train trajectories must be described with the inclusion of tractive efforts data [21] and through motion phases specification, i.e. acceleration, cruise, coasting, braking [22, 23].

As previously said, data collected onboard are often affected by errors. In some cases, such as in train trajectories collection, errors on the single data source are almost unavoidable (i.e. train running into tunnels, in canyons, etc.) and only (ex-post) reconstruction procedures may help estimating the characteristics of train operation. Formally, the train trajectory reconstruction is the estimation, with a minimum error, of the position, velocity, acceleration of train vehicles, for a sequence of time points, along a digital infrastructure model. This practice has been faced in the last years in different fields of transport systems [7, 25, 26, 27], with the increasing amount of data available that come from different sources. In railway systems, main studies refer to data from Global Navigation Satellite Systems (GNSSs), odometers and Inertial Measurement Units (IMU). GNSSs in particular are taking even more interests in the field; currently their use is allowed only for non-safety applications, but future signaling systems, i.e. moving block, will be based on them [37]. Main reason relies in the possibility to dismantle the track equipment dedicated to train position with evident, consequent cost reduction. Main current research trends are based on advances in data fusion techniques between the abovementioned data sources and on algorithms, to reduce train-positioning error [11, 12, 13, 14, 15, 16]. Train trajectories can be reconstructed also with the help of additional information sources such as track occupation data [1]. Alternative approaches such as map matching algorithms [3, 8, 24], can be also considered. These last are able to associate multiple GPS points to the links of a network, in order to estimate routes and trajectories travelled by vehicles. However all these techniques are linked with technologies that still need some track equipment for calibrating/realigning themselves (e.g. odometers with balises), therefore, their use can be limited in the future.
The use of filtering techniques, such as those used for online train trajectory estimation has been also explored. This has led to the development of several recursive linear estimators for train position, such as Kalman Filters (KF) or Extended Kalman Filters (EKF) [6, 9], or nonlinear estimators based on Particle Filter (PF) [4, 5]. However, these methods return suboptimal solutions in case of trajectory reconstruction (offline procedure) since the solution is based on the last data collected to determine a current value, and not on the entire trajectory data.

Currently, the literature lacks of in depth systematic analyses that help practitioners and researcher have a comprehensive foundation on train trajectory reconstruction. In authors’ opinion, future models for traffic management and train control will make an extensive use of train trajectories. Reconstruction procedures will be more and more used to minimize measurement errors and to increase the congruence between different data sources. The future, in authors’ opinion, will see the rise of data-driven approaches, commercially known as Big Data, as a competitive alternative to model based frameworks (de Martinis and Corman, in press). Trajectory reconstruction will be a fundamental practice; the aim of this paper is therefore to contribute to the relatively young discussion with a novel hybrid model that fits current and future standards.

This paper aims also at exploring the possibility to reconstruct train trajectories through onboard-only measurements. In other words, all track based equipment (e.g. electric circuits) or onboard equipment that use on-track devices for realignment (e.g. odometers) are not considered. To do this, a novel approach based on both kinematic (position, velocity, acceleration) and dynamic (power used for traction, resistances) information from traction unit is proposed.

In conclusion, the main innovative contributions to the field are:

- A new optimization algorithm for train trajectory reconstruction, based on the criterion of batch Maximum a Posteriori (MAP) smoothing [30], which combines dynamics and kinematic characteristics of the train. To the best of authors’ knowledge, the here proposed combination of dynamics and kinematic information to reconstruct the most likely trajectory has never been considered in the field and it can enhance the quality of the estimates.

- Inclusion of measurements stochasticity into trajectory reconstruction process. We evaluate the robustness of the reconstructed trajectories by adding different values of measurements noise. In the present work, a special case study is dedicated to this aspect; white noise of different magnitudes is added to velocity measurements and the model response is evaluated accordingly.

The rest of the paper is organized as follows. In Section 2 we introduce the two main descriptions of motion, i.e. dynamics and kinematics, and we specify it for trains. In Section 3 we propose a trajectory reconstruction algorithm based on the criterion of batch MAP (Maximum A Posteriori) smoothing, which include both the two aforementioned formulations. In Section 4, the proposed algorithm is implemented and evaluated over multiple train runs through three different case studies. Section 5 summarizes the experiences gained with the case studies and conclude the discussion, highlighting further developments and investigations.
2 Train motion modeling

This section describes the train motion models supporting the trajectory reconstruction algorithm. In the next subsections, we describe two train motion models that refer respectively to kinematics and to dynamics approaches. Kinematics-based models describe the motion of an object uniquely through motion variables, i.e. acceleration, velocity, position. The description of the motion of an object is what we call trajectory. The geometric nature of its formulation allows simplifying the derivation of the equations of motion. Dynamics-based models link the effect of forces and torques applied to an object to the motion of the object itself. The final output is still a trajectory as in kinematic models, but more input data are needed and differential equations must be solved. On the other hand, trajectories from dynamic models are often more realistic. Dynamic models are mostly derived from Newton’s second law of motion.

2.1 Kinematic representation of train motion

Kinematic models of train motion mainly consider variables derived from the positions and time. Typically, the train motion characteristics are described through a set of variables \( s_k, v_k, \) and \( a_k \) representing curvilinear position, velocity, and acceleration of the train on the track at time \( k \). For our purposes, we adopted a Discrete-time Weiner Process Acceleration model (DWPA) as reference model \([32]\). The general formulation is the following:

\[
\begin{bmatrix}
S_{k+1} \\
V_{k+1} \\
A_{k+1}
\end{bmatrix} =
\begin{bmatrix}
1 & \Delta t & \frac{\Delta t^2}{2} \\
0 & 1 & \Delta t \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
S_k \\
V_k \\
A_k
\end{bmatrix} +
\begin{bmatrix}
\frac{\Delta t^2}{2} \\
\Delta t \\
1
\end{bmatrix}
\delta_k ,
\]

where \( \Delta t = (k + 1) - k \) is the sampling time and \( \delta_k \) is a white noise process symbolizing the acceleration variation. The noise \( \delta_k \) is assumed Gaussian with zero mean and standard deviation \( \sigma_\delta \). The goodness of this model is overall simplicity and the inclusion of additional factors such as noises representing possible variation from general assumptions. On the other hand, there is no possibility to constrain variables to their feasible values, e.g. acceleration varies with velocity and it depends on several factors such as train features and infrastructure characteristics.

2.2 Dynamic representation of train motion

Dynamics considers forces applied to the vehicle and associates motion characteristics. The commonly used dynamic model of motion is expressed through differential equations derived from Newton’s second law. Compared to kinematics formulation, train motion characteristics therefore include also information on tractive efforts (or alternatively power used), braking efforts and resistances given both by infrastructure characteristics (e.g. gradients, curves) and by vehicle characteristics (e.g. weight, aerodynamics).

For our purpose, we consider the widely used mass-point dynamic model for the train (see \([33]\) for a more extensive description)

\[
T_k = m \ a_k + R(v_k) + m \ g \sin \gamma_k + mg \ 700/r ,
\]
with \( T_k \) being the tractive/braking efforts at train’s wheels, \( m \) the mass of the train. We model the vehicle resistances \( R(v_k) \) with the widely used trinomial formulation specified during the decades by different authors [34]:

\[
R(v_k) = r_0 + r_1 v_k + r_2 (v_k)^2.
\]

(3)

It is worth to remind that tunnel resistances are usually considered as an additional term of the quadratic parameter of (3) (see [34] for more details), although this resistance is activated at specific positions of the train on the track. In our study, the coefficients’ values \( r_0, r_1, r_2 \) have been given by the train operator.

Line resistances are here intended as resistances given by gradients and curves. Gradient resistances are expressed with \( mg \sin \gamma_k \). The term \( \gamma_k \) represents the gradient of the point on track the train is positioned at time \( k \). Note that in case of descent track, this parameter is negative and the whole term gives a concordant contribution to the train motion. The term \( mg 700/r \) represents the curve resistances, \( r \) is the radius of the curve. In the following sections, curve resistances have been neglected to simplify the formulation, since only large radii are present in following case studies.

The acceleration rate is limited by the minimum value between the one generated by the maximum tractive efforts \( T_{\text{MAX}}^k \) that can be produced by the traction unit, and the other representing the adhesion limits \( Ad \).

\[
a_k = \min( a(T_{\text{MAX}}^k), a(Ad) )
\]

The adhesion limits can be computed with the formulation of Curtius and Kniffler [36]. It is worth noting that the desired acceleration (i.e. from train driver’s behavior) can be lower than the maximum acceleration allowed by the traction unit and the infrastructure geometry; in this case, the corresponding tractive efforts are a result of eq. (2). Maximum tractive efforts are usually known and expressed by the tractive efforts-velocity diagram, which is provided by the train builder.

3 Trajectories reconstruction through batch MAP smoothing

The scope of the present section is to describe the proposed model for trajectory reconstruction, the related assumptions and the data supporting the whole process. Given a history of recorded on-board measurements on the characteristics of train motion (e.g. position, velocity, power used), the main aim of the trajectory reconstruction model here presented is to estimate the trajectory followed by the train between two consecutive planned stops, so that the values of the train motion variables are congruent and consistent along time. Congruent is here intended as something formally verified by mathematical relations, e.g. the relation between velocity, space and time. The term consistent here refers to a dataset without any missing value or errors. Assuming that we know the route of the train, the reconstructed trajectory consists of estimating positions, velocities and accelerations of the train on the track, as a function of time. Among all the possible trajectories, we select the one that agrees the most, in a probabilistic meaning, with the history of recorded measurements affected by error and missing data and with the train motion models described in the previous section. More specifically, the problem of train trajectory reconstruction is here formulated as a batch Maximum A Posteriori (MAP) smoothing problem [27, 29, 30].
The reconstructed trajectory is obtained by fusing the on-board recorded data with the train motion models and solving a constrained Least Squares optimization problem. The rest of the section is organized in three subsections. In the first subsection we introduce discrete-time process equation and measurements equation describing the train motion and relating it to the on-board measurements, respectively. In the second subsection, trajectory reconstruction is formulated as a batch MAP smoothing problem that considers both process equations and measurement equations. In the last subsection, a constrained Least Squares optimization quadratic program is proposed for the reconstruction of train trajectories from recorded data, accordingly to the batch MAP criterion.

### 3.1 Process and measurement equations for train motion

The process equation describes the discrete-time evolution of the state of the train, here intended as train position, velocity and acceleration. To simplify the notation, we denote with $x_k = [s_k, v_k, a_k]^T$ the state of the train at time $k$. Moreover, we define the matrices:

$$
A = \begin{bmatrix}
1 & \Delta t & \frac{\Delta t^2}{2} \\
0 & 1 & \Delta t \\
0 & 0 & 1 \\
\end{bmatrix} ;
G = \begin{bmatrix}
\Delta t \\
\Delta t^2 \\
1 \\
\end{bmatrix}
$$

Hence, according to the kinematic model (1), we obtain the Process equation:

$$
x_{k+1} = Ax_k + w_k
$$

with $w_k$ Gaussian zero-mean, independent over time, and with covariance matrix $Q_k = G^T G \sigma_w^2$. We refer to $w_k$ as process noise.

The measurements equation relates the state of the train $x_k$ to the on-board recorded measurements. Moreover, it includes the presence of additive measurement noise proper of each sensor. The on-board measurements consist of GPS position, velocity, and consumed active power (measured at pantograph). By using the digital track topology, we decided to project each GPS measurement onto the digital track map. In fact, the track followed by the train is assumed being know at each time step, and given as a piecewise linear curve in the plane. In this manner, curvilinear position measurements denoted with $y_k^{(1)}$ are obtained. Due to the typically large radii of curvature, the error committed considering $y_k^{(1)}$ instead of the original GPS measurements is assumed negligible in a maximum a posteriori sense. This assumption allows to consider a simpler and linear kinematic model (1) with the train moving on a 1D coordinate system. Moreover, we let $y_k^{(2)}$ be the velocity measurements recorded at time $k$. Measurements equations for position and velocity thus result:

$$
y_k^{(1)} = s_k + n_k^{(1)}
$$

$$
y_k^{(2)} = v_k + n_k^{(2)}
$$

with $n_k^{(1)}$, $n_k^{(2)}$ being the associated measurement noises. Similarly, we let $y_k^{(3)}$ be the measured active power. Note that it can be strictly related to the traction at wheels as:
\begin{align}
\gamma_k^{(3)} &= \frac{1}{\eta} T_k v_k + n_k^{(3)}, \quad (7)
\end{align}

where the constant $\eta \in (0,1)$ represents the energy losses relating the power measured at the pantograph to the one transmitted at wheels, and $n_k^{(3)}$ is the power measurement noise. Hence, using (7) and (2) we have

\begin{align}
\gamma_k^{(3)} &= \frac{v_k}{\eta} (m a_k + R(v_k) + m g \sin \gamma_k) + n_k^{(3)}. \quad (8)
\end{align}

Strictly speaking, the gradient $\gamma_k$ is a function of the train position (i.e. $\gamma_k = \gamma(s_k)$). However, due to its small variations along the track we approximate it as function of the projected GPS measurement, i.e., $\gamma_k = \gamma(y_k^{(1)})$. In case of missing measurements, we select $\gamma_k$ assuming the train moved with constant acceleration in the interval where measurements are not available.

At this point, we let $\gamma_k = \left[\gamma_k^{(1)} \gamma_k^{(2)} \gamma_k^{(3)}\right]^T$ be the measurements vector a time $k$, and $n_k = \left[n_k^{(1)} n_k^{(2)} n_k^{(3)}\right]^T$ the overall measurement noise.

Defining

\begin{align}
h_k(x_k) := \begin{bmatrix} s_k \\ v_k \\ \frac{v_k}{\eta} (m a_k + R(v_k) + m g \sin \gamma_k) \end{bmatrix}, \quad (9)
\end{align}

and combining (5),(6), and (8) we obtain the following Measurements equation:

\begin{align}
\gamma_k = h_k(x_k) + n_k \quad (10)
\end{align}

We assume $n_k$ is Gaussian zero-mean, with covariance matrix $R_k = \text{diag}(\sigma^2_{\text{pos}}, \sigma^2_{\text{vel}}, \sigma^2_{\text{pow}})$. In case only a subset of position, velocity, and active power measurements is available at time $k$, the dimension of $\gamma_k$ (and of $h_k(\cdot)$) is tacitly reduced.

We let $x_0$ be the train’s initial state (at departure station) and assume it is normally distributed with mean $\bar{x}_0$ and covariance matrix $P_0$. Typically, a precise knowledge of $x_0$ is available (e.g. from timetable information), therefore $P_0$ can be set very small. Moreover, we let $N$ be the time duration of the train trajectory that we want to reconstruct. The problem of trajectory reconstruction, hence, consists of finding an estimate of the sequence $\{x_0, \ldots, x_N\}$, given the history of recorded measurements $\{y_1, \ldots, y_N\}$. We will denote such estimate by $\{\hat{x}_0, \ldots, \hat{x}_N\}$.

### 3.2 Batch Maximum A Posteriori (MAP) smoothing

Among all the possible trajectories, the batch MAP smoothing selects the one which maximizes the posterior joint probability of $\{x_0, \ldots, x_N\}$, conditioned on the history of measurements $\{y_1, \ldots, y_N\}$. That is,

\begin{align}
\{\hat{x}_0, \ldots, \hat{x}_N\}_{\text{MAP}} = \arg\max_{x_0, \ldots, x_N} p(x_0, \ldots, x_N | y_1, \ldots, y_N). \quad (11)
\end{align}

The MAP trajectory (11) **Error! Reference source not found.** can be obtained by the following chain of equalities:
\[
\{\hat{x}_0, \ldots, \hat{x}_N\}_{MAP} = \arg \max_{\{x_0, \ldots, x_N\}} p(x_0, \ldots, x_N | y_1, \ldots, y_N) = \arg \max_{x_0 \rightarrow x_N} p(y_1, \ldots, y_N | x_0, \ldots, x_N) p(x_0, \ldots, x_N) = \arg \max_{x_0 \rightarrow x_N} p(x_0) \prod_{k=1}^{N} p(y_k|x_k) \prod_{k=0}^{N-1} p(x_{k+1}|x_k) = \arg \min_{x_0 \rightarrow x_N} -\log p(x_0) + \sum_{k=1}^{N} -\log p(y_k|x_k) + \sum_{k=0}^{N-1} -\log p(x_{k+1}|x_k) = \arg \min_{x_0 \rightarrow x_N} \|x_0 - \bar{x}_0\|_Q^{-1} + \sum_{k=1}^{N} \|y_k - h_k(x_k)\|_R^{-1} + \sum_{k=0}^{N-1} \|x_{k+1} - A x_k\|_Q^{-1} (12)
\]

The first equality follows from Bayes rule, the second from independence of process and measurement noises, the third from monotonicity of \(\log()\), and the last by the assumption of Gaussian initial state \(x_0\), process noises \(w_k\), and measurement noises \(n_k\), and by the process and measurements equations introduced in Section 3.1. For a complete derivation of (12), please refer to [30].

The batch MAP estimate \(\{\hat{x}_0, \ldots, \hat{x}_N\}_{MAP}\), hence, corresponds to a Least Squares solution. More specifically it minimizes the deviations from the expected initial state \(\hat{x}_0\), the measurements \(y_k\), and the model mismatches. Each of these terms is weighted by the associated covariance matrices \(P_0, R_k,\) and \(Q_k\), respectively.

### 3.3 A Least Squares optimization problem for train trajectory reconstruction

As previously said, MAP estimates can be obtained by formulating and solving a Least Squares optimization problem. Moreover, using process and measurements equations (4) and (10), it is possible to write the Least Squares expression of (12) more explicitly. In particular, the equality constrained Least Squares optimization problem [29, 30] can be expressed in the following way:

\[
\{\hat{x}_0, \ldots, \hat{x}_N\}_{MAP} = \arg \min_{x_0 \rightarrow x_N} \|x_0 - \bar{x}_0\|_Q^{-1} + \sum_{k=1}^{N} \|n_k\|_R^{-1} + \sum_{k=0}^{N-1} \|w_k\|_Q^{-1} (13)
\]

s.t. \(x_{k+1} = A x_k + w_k \quad k = 0, \ldots, N - 1\)
\(y_k = h_k(x_k) + n_k \quad k = 1, \ldots, N\).

Based on eq. (13)#(13), we here propose a Least Squares optimization problem for train trajectory reconstruction.

The state \(x_k\) of the train comprehends only kinematic variables \(s_k, v_k,\) and \(a_k\). Therefore, we can enforce additional knowledge to the batch MAP smoothing problem, by constraining the estimated trajectory to lie in a feasible operational region for the train. This is done adding extra inequality constraints. In this way, estimated state variables are strictly related to the physical characteristics of the train and to the limits allowed by the infrastructure. Such constraints are introduced in the next paragraph. They comprehend maximum allowed velocities on the track, maximum deceleration to guarantee passengers comfort, and maximum traction at wheels defined by the train’s traction vs. velocity curve at maximum power. As discussed in [31], whether these additional constraints preserve the MAP interpretation in the optimization problem is more a question of philosophy than statistics. Nevertheless, they have practical relevance and typically improve the smoothing algorithm.
The proposed Least Squares optimization problem for train trajectory reconstruction consists of eq. (13), together with the following additional constraints. The velocities are constrained as:

\[ 0 \leq v_k \leq \bar{V}_k \quad k = 0, \ldots, N \]  

(14)

with \( \bar{V}_k \) being the maximum velocity allowed in the track section at time \( k \). Strictly speaking, \( \bar{V}_k \) is a function of the position \( s_k \). However, we can adopt similar arguments as for the slopes \( y_k \) (in Section 3.1). In this case, when two different values for \( \bar{V}_k \) are plausible given the position measurement \( y_k^{(1)} \), we choose \( \bar{V}_k \) to be the highest of the two values. Moreover, we constrain the accelerations \( a_k \) as

\[ a \leq a_k \leq \frac{1}{m} (\bar{T}(v_k) - R(v_k) - m \, g \sin y_k) \quad k = 0, \ldots, N \]  

(15)

with \([a]\) being the maximum allowed deceleration (for passengers comfort), and \(\bar{T}(v_k)\) being the train’s traction vs. velocity curve at maximum power. \(\bar{T}(v_k)\) is modeled as

\[
\bar{T}(v_k) = \min \left\{ \frac{c}{v_k^2} \right\}, \quad 0 < v_k < v_{\text{max}},
\]

where \( T_{\text{max}} \) and \( c \) are parameters given by the train manufacturer, and \( v_{\text{max}} \) can be assumed beyond the allowed maximum velocities. Using (16), constraint (15) can be explicitly rewritten as:

\[ a \leq a_k \leq \frac{1}{m} (T_{\text{max}} - R(v_k) - m \, g \sin y_k) \quad k = 0, \ldots, N \]  

(17)

\[ a \leq a_k \leq \frac{1}{m} \left( \frac{c}{v_k^2} - R(v_k) - m \, g \sin y_k \right) \quad k = 0, \ldots, N \]

To summarize, we reconstruct past train trajectories of duration \( N \), by solving:

\[
\arg\min_{x_0, \ldots, x_N} \left\| x_0 - \tilde{x}_0 \right\|^2_0 + \sum_{k=1}^{N} \left\| n_k \right\|^2_{P^{-1}} + \sum_{k=0}^{N-1} \left\| w_k \right\|^2_{Q^{-1}}
\]

s.t. \[ x_{k+1} = A x_k + w_k \quad k = 0, \ldots, N - 1 \]
\[ y_k = h_k(x_k) + n_k \quad k = 1, \ldots, N \]

equation (9)

equation (17)

The proposed optimization problem (18) has a quadratic objective function, nonlinear equality constraints and nonlinear inequality constraints. The nonlinearities arise from the power measurement equation (8) and from the maximum traction constraints (right hand sides of (17)), while the rest of the constraints are all linear.

In order to efficiently solve (18), we propose the following two-step procedure:

1) Solve a simpler version of (18), which does not use power measurements and neglect the maximum tractive power constraints. We denote such program with the attribute no Power. Differently from (18), all the constraints of the problem no Power are linear and it results in a simpler quadratic program for which off the shelf solvers can be used (e.g., Matlab solver QUADPROG). Note that the so-estimated trajectory does not take into account train dynamic information.

2) Solve a linearized version of (18), where the power measurements equation (8) and the right hand side of constraints (17) are linearized around the trajectory estimated at step 1). We denote such program with the attribute with Power. Having linearized (18), off-the-shelf quadratic programming solvers can be used also in this case.
Following the above-mentioned two step procedure and solving the linearized version of (18), approximate solutions are therefore obtained. The goodness of such linearization is empirically verified over multiple train runs. In the next section, the proposed algorithm (18) is implemented and evaluated for the reconstruction of multiple train trajectories through three different case studies.

4 Case Studies

4.1 Experimental Setup

We consider multiple runs between two stops at stations of a passenger train on a Swiss line, with an average duration of 1024 seconds. The track is approximately 37 km long and digitally provided as a piecewise linear curve in the LV95 Swiss coordinate system. Four tunnels are present of lengths 0.35, 0.27, 1.25, and 2.23 km, respectively. Recorded sensors data consist of GPS position, velocity, and consumed active power. In proximity of tunnels, GPS and velocity measurements are typically missing or are highly inaccurate. For this reasons, they have been discarded from 3 seconds before the entrance up to 3 seconds after the exit of each tunnel. Consumed active power, however, was constantly measured each second. The maximum allowed velocity on the line is 160 km/h (∼ 44.4 m/s).

The train operator provided us with the mass $m$ of the train and the resistances parameters $r_0, r_1, r_2$. The constant $\eta$ has been set to 85\% symbolizing transmission and conversion losses [35]. The traction curve at maximum power is defined by $T_{\text{max}} = 240$ kN and $c = 5.2 \cdot 10^6$ W·s/m. The maximum allowed deceleration $g$ has been set to $-0.8$ m/s$^2$. For the maximum velocities $\bar{v}_k$, a tolerance of 0.5\% is considered.

For the introduced discrete-time models, we chose a sampling time $\Delta t = 1$ s. The following values were selected as measurement noise standard deviations. Assuming the GPS error is a 2D Gaussian with covariance matrix diag($\sigma_{\text{GPS}}^2, \sigma_{\text{GPS}}^2$), one can verify that $\sigma_{\text{pos}} = \sigma_{\text{GPS}}$ , being $n_k^{(1)}$ the projection of the 2D GPS error along the track. We empirically estimated $\sigma_{\text{GPS}}$ to be 1.39 m computing the distances between GPS measurements and the digital track. Then, we selected $\sigma_{\text{pos}} = 1.5$ m. For the velocity measurements, we selected $\sigma_{\text{vel}} = 0.1$ m/s. Finally, due to the known inaccuracies of the introduced dynamic model and of the efficiency $\eta$, we chose $\sigma_{\text{pow}} = 0.1y_k^{(3)} + 10^6$ W. The measured positions $y_k^{(3)}$ are taken from the track starting point (0 m). For the process noise, we selected standard deviation $\sigma_{\delta} = 0.2$ m/s$^2$

In the following three case studies, we evaluate the performances of the proposed optimization problem (18) for trajectory reconstruction, showing the goodness of the introduced dynamic-kinematic model for the train. The case studies also show the advantages of combining kinematic measurements with the dynamic information about power consumption. The latter is demonstrated to be a valuable data source for trajectory reconstruction, especially when conventional sources are missing or are not reliable (e.g. in tunnels). All the computations are carried on MATLAB 2017b, installed on an Intel i7 machine @ 3.7 GHz with 16 GB of RAM. The optimization problems are solved by the QUADPROG solver, through Yalmip.

4.2 Case study 1: multiple train trajectories

For this first case study, we consider more than 250 runs of identical trains running on the considered track. For each run, the proposed optimization problem formulated
in eq. (18) is solved and an estimated trajectory is obtained, as outlined in the previous section. The problem is solved under two conditions: in no Power condition a simpler version with no power measurements is proposed; in with Power conditions, a linearized version is considered, as described in steps 1) - 2) of Section 4.3. Due to unavailability of ground-truth data, the goodness of the reconstructed trajectories is evaluated as follows.

1) We feed the algorithm with power measured at each sampling time, but with downsampled position and velocity measurements. By a down-sampling gap of $X$ seconds, we mean that position and velocity measurements are available to the algorithm each $X$ seconds. 2) We evaluate the absolute error between the estimated profiles (position vs. time and velocity vs. time) and the original measured ones.

To evaluate the goodness of the introduced dynamic models and the added value of data about consumed active power, we also compare the reconstructed trajectories in with Power and no Power conditions.

Figure 1 shows the absolute errors between the reconstructed and measured velocity profiles of all the train runs, as a function of the measured positions $y_k^{(1)}$, for different

![Figure 1](image1.png)

Figure 1: Absolute position and velocity errors, for increasing down-sampling gaps, with and without using power information. Medians (red bars), 25th and 75th percentiles (in blue).

![Figure 2](image2.png)

Figure 2: Absolute velocity errors as a function of the estimated positions on the track, for increasing down-sampling gaps, with and without using power information.
down-sampling gaps. Figure 2 shows the boxplots of the absolute errors in position and velocity, for all the train runs. As expected, the errors grow with the increase of the down-sampling gap, since less measurements are available to the algorithm. However, Figures 1-2 show how the proposed algorithm with Power steers the estimates significantly closer to the original measured profiles than when its simpler version no Power is used. Hence, the power measurements and the dynamic characteristics of the train (Section 3) allow the proposed algorithm with Power to achieve more realistic estimates.

Figure 3 shows the absolute linearization errors as a function of the down-sampling gaps. For each reconstructed trajectory, and at each time step $k$, the linearization error is computed as the difference between the right hand side of the power measurement equation (8) and its linearized version (Section 4.3 step 2), evaluated at the estimated state $\hat{x}_k = [\hat{s}_k, \hat{v}_k, \hat{a}_k]^T$. Relative linearization errors are also reported, as the ratio between the absolute linearization errors and the absolute value of the measured powers. As expected, such errors increase with the increase of the down-sampling gap, since the algorithm no Power provides an always less accurate trajectory around which to linearize. As visible, however, especially for down-sampling gaps of 1-30 seconds, the errors are very small (< 0.8% of the measured power). For completeness, the CPU times of solving both the quadratic programs are reported in Figure 4. As expected, they are decreasing when less data are available to the algorithm (increasing down-sampling gap or not considering data about power consumption).
4.3 Case study 2: missing recorded data in the presence of tunnels

In this second case study, we consider a single train run and we focus on the central portion of the considered track, where the four tunnels are present and the train typically reaches the maximum allowed velocity (160 km/h). This case study intends to give the reader a perception of the model’s capabilities when missing or unreliable data, such as those in presence of tunnels, are present. It is clear that the most natural field of evaluation for this type of experiment is a tunnel; however, this is also the limitation of this case study because this also means that ground truth is not collected, i.e. there are no measurements for comparison.

To highlight the goodness of the proposed algorithm and keep the tunnel as field of evaluation, we decided to ‘hide’ the GPS and velocity measurements collected between two consecutive tunnels from the beginning of the first tunnel, up to the exit of the fourth one, thus creating a fake long tunnel for a total length of 7.7 km. We evaluate, then, how close is the reconstructed trajectory to the hidden measurements.

In Figure 5, the upper graph shows the velocity profile reconstructed by the proposed algorithm with Power as well as the one by its simpler version no Power. Due to measurements unavailability, the algorithm no Power sees no recorded data for a time

![Graph showing velocity profile reconstruction](image)

![Graph showing power consumption comparison](image)

Figure 3: Top: measured vs. reconstructed velocity profiles with and without using power. Middle: measured vs. reconstructed power consumptions with and without using power information. Bottom: gradient profile.
window of about 200 s. In such window, the trajectory estimated by no Power simply tries to match initial and final velocities, as well as traveled distance, using the smoothest possible acceleration profile. The proposed algorithm with Power, however, uses the extra information about consumed active power and its estimates are very close to the (hidden) measured velocities.

Figure 5 also shows the measured power $\eta \cdot y_k^{(3)}$ (transmitted at wheels) and the estimated power $P_{\text{hat}} = \hat{v}_k (m \hat{a}_k - R(\hat{v}_k) - m g \sin \gamma_k)$, where $\hat{v}_k$ and $\hat{a}_k$ are the estimated velocities and acceleration. As visible, in the time steps where measurements were hidden (around 300 to 500 s), the power estimated by the algorithm no Power mostly matches the gradient profile, while the velocity and acceleration profiles reconstructed by the proposed algorithm with Power match the power consumption. This shows the goodness of the introduced dynamical model for the train and identify the active power as a valuable source of information, especially when other sources are missing or are not reliable.

4.4 Case study 3: algorithm response to perturbed measurements

In this third case study, we consider the same train runs of Case Study 1 and evaluate how the proposed algorithm responds to perturbed velocity measurements. We believe that the recorded velocities are accurate enough to be considered as ground truth for the case study, and feed the algorithm with perturbed values of these. In particular, we perturb each of the velocity profiles with additive zero-mean normally distributed noise with standard deviations $\sigma_{\text{pert}} = 0.5$ m/s, 1 m/s, and 2 m/s. Then, we evaluate the adherence of the reconstructed velocity profiles to the measured ones. As in the previous cases we refer to two different algorithms, with Power and no Power, that respectively consider and not consider information on power used. The parameters for both the algorithms, consisting of standard deviations both for process and for measurement noises, are set up as in Section 4.1, except for the standard deviation $\sigma_{\text{vel}}$. The parameter $\sigma_{\text{vel}}$ allows to analyze two main scenarios. In the first scenario, we select $\sigma_{\text{vel}}$ to be equal to the standard deviation $\sigma_{\text{pert}}$. This simulates the case where accurate information about sensors’ noise are available and/or can be accurately estimated. In this scenario, we assume that the measurement system has an adaptive behavior, i.e. it recognizes and estimates the noise magnitude, thus it modifies the $\sigma_{\text{vel}}$ from case to case. In the second scenario, we fix $\sigma_{\text{vel}} = 0.1$ m/s (as in Section 4.1), regardless of the perturbation noise. More realistically, in fact, statistics about sensor noises are not available or can be inaccurate. Hence, in the second scenario, the proposed algorithm underestimates the velocity measurements noise.

Figure 6 shows the absolute errors between the original measured velocities and the ones reconstructed by the two algorithms, as a function of the standard deviation $\sigma_{\text{pert}}$ of the perturbation noise, for both the aforementioned scenarios. As expected, when accurate noise statistics are available to the algorithm (first scenario, left plot of Figure 6), the reconstructed velocities are robust against the added noise and the errors are kept low, regardless the magnitude of the perturbation. However, when the algorithm underestimates the noise standard deviation (second scenario, right plot of Figure 6), such errors increase with the increasing of the perturbations. In both the scenarios, as visible from the figure, the extra information of consumed active power, maintains the error lower than when only velocity and position measurements are available (no Power). Data about power consumption, in this case, help the algorithm to reject the added noise and provide more robust estimates.
Final discussion and future work

In the present paper, a novel hybrid approach for train trajectory reconstruction based on GPS data and power measurements at traction unit is presented. Train operation data are a key information for offline elaboration, e.g. timetable generation, driving behavior analysis, models’ calibration. Unfortunately, the collection of these data is often affected by errors, missing measurements, noises. To enhance rail operation performances (e.g. more trains per hour, increase of energy efficiency), these data need to be more precise, reliable and detailed. The proposed approach allows solving some typical problems linked with data on train trajectories, such as missing GPS measures in tunnels, thanks to onboard measures on power consumption at traction unit. This results in a hybrid formulation of a MAP (Maximum A Posteriori) problem where kinematics and dynamics formulations are considered together. Kinematic models only describe the motion of an object; the description of the motion of an object is also called trajectory. The geometric nature of its formulation allows simplifying the derivation of the equations of motion. Dynamic models are based on the interaction of forces and torques with the motion of an object. The final output is still a trajectory as in kinematic models, but more input data are needed and differential equations must be solved. On the other hand, trajectories from dynamic models are linked with the forces applied, i.e. tractive efforts and resistances; this link enables a better fitting to real motion. The hybrid approach here proposed combines the positive aspects of the two models; it keeps a simple formulation of the motion and it adds an important source of information; i.e. the tractive efforts applied, computed from measurements on power used, and the resistances. Another important advantage lies in the use of onboard-only set of measurements, thus completely released from any track-based equipment. This aspect is very relevant for applications with future signaling technologies because the trend is to completely dismantle track-based equipment and therefore to decrease operative costs. Results have shown that the additional information on power used gives interesting results in terms of reconstruction accuracy. The reconstructed trajectories with the proposed approach are, among all possible trajectories, the ones that had better followed the power used and, consequently, the tractive effort applied.

Figure 6: Left: absolute velocity errors when the exact noise statistics are available $\sigma_{vel} = \sigma_{pert}$. Right: absolute velocity errors when noise statistics are not available $\sigma_{vel} = 0.1 \text{ m/s}$. Medians (red bars), 25th and 75th percentiles (in blue).
Compared with results from trajectory reconstruction without considering power information, the model returns smaller errors on position and velocity estimation. This result is even more evident when considering possible missing data or errors in data collection; by increasing the gap of missing data up to 120 seconds, the proposed approach returns errors that not only are smaller but also have a slower growth.

In case of tunnel, the proposed model returns very good results. The shape of the reconstructed trajectory with the proposed approach is not only compatible with the power profile but also, as shown in our tests, fits some velocity measurements that have not been presented to the algorithm.

In case of perturbed measurements, the extra information of power used and resistances keeps the error smaller than when only velocity and position measurements are available. This result is confirmed both in case of an adaptive configuration of our model, i.e. it is possible to estimate the magnitude of perturbation from time to time, and in case of a static configuration, i.e. by considering a predefined level of perturbation. In the first case, results are obvious; the model gives less importance to the measurements when it knows that these are more perturbed. In the second case, it is more evident how information on power used allows reconstructing the train trajectories with smaller errors on the velocity values taken as reference for ground truth.

In the end, the proposed approach results a promising road for train trajectory reconstruction, with increased precision and in onboard-only measurements conditions, and fully compatible with the future trends and need of next-generation traffic management systems.

Further steps in this research trends are seen in the tuning and in the validation of a model based on the proposed hybrid stochastic approach. The first trend mainly refers to the parameters values here assumed; although the already good results of the model, a better parameters tuning, for example, of the power efficiency from pantograph to the traction unit and of the resistance can further enhance the model’s performances. The other trends refer to the model validation with data that can be considered as ground truth. In authors’ opinion, the identification of a ground truth in this field represents a hot challenge also for other related research fields (e.g. signaling systems, traffic management) and a task that can be fulfilled only with specific campaigns of data collection organized with the help of train operators and infrastructure managers. Moreover, the complexity and the computing time for trajectory reconstruction increases proportionally to the time duration of the trajectory. Although this is an offline procedure, further investigations with alternative algorithms and decomposition techniques may return faster responses thus enabling the procedure to be considered for a wider range of applications.

References