Using Continuous Approximation for Public Transport Service Quality and Fare Level Optimisation

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1. Introduction

1.1. Background
Prices for public transport vary significantly across different parts of the world. Fare increases are often a complex political issue as the prices are in many cases at least to some degree regulated. Providing affordable public transport is associated with a very wide range of issues, including urban development, congestion regulation, environmental issues as well as general ideas of fairness and providing a basic mobility level for all. Whereas in many developing countries the fare is kept low in order to allow people with a diverse financial background being able to afford it, in other countries the idea that public transport users should pay for the costs their travel creates leads to significantly higher prices. Furthermore, higher fares are often justified with the argument that it allows providing a better service. Due to these issues and different objectives in general there appears to be little agreement as to what fares are appropriate or optimal. In a recent report Schmöcker et al (2016) also describe that the definition of “fair fares” varies across cities and transport authorities. Whereas some cities consider highly differentiated fares (by distance, user group and/or time) as fair, others consider simple fare structures as fair.

The objective of this paper is to contribute to this discussion by showing how fares, at least under a number of simplifying assumptions, would lead to different service quality levels and with it different demand levels. We consider that the answer to this will to a large part depend on the city parameters. We aim to provide some guidance as to what fares; PT service quality and demand levels can be derived in a range of cities. More specifically as input and city parameters we vary size, population density and the demand level. Building on the work of Daganzo (2010), the variables that we vary are optimal network headway, stop spacing as well as the ratio of a central dense PT service area compared to the whole city size. These variables together with the fare level and the demand, which we presume to be elastic, interact and provide us with indices of social welfare and subsidy needed for the operator.
The remainder of this paper is organised as follows. In the next section, existing literature is reviewed and in Section 3, the equations used in the model are explained. Section 4 compares the sensitivities of the optima regarding the decision variable boundaries and input variables, to observe how the differences are interrelated. Finally, Section 5 will conclude including further work directions.

2. Literature Review

The vast majority of previous research on fares and pricing is either qualitative or on congestion pricing for traffic. As a study looking at general networks we note here the study by Tikoudis on congestion pricing (2015). The insights that can be gained from his study for public transport pricing are though limited as it neglects some of the public transport characteristics. For example higher fares can lead to the operator having more resources to provide more frequent and possibly more varied services leading to less waiting and less transfers and/or walking requires. Also qualitative aspects of the service might improve such as vehicle cleanliness or safety which are also often key determinants of travel demand.

Existing quantitative research on public transport fares are mostly case studies, or are rather emphasizing operators’ aspects. For example, Beirao et al. (2007) conducted a qualitative study on attitudes towards public transport and private car. The study of Li et al. (2009) compared the efficiency of different transit market regimes in transit services that are not reliable. Zhi-Chun Li et al. (2008) did propose a fare structure optimization model and focused on operators in different market regimes. Rough relationships between revenues, ridership and fare levels are introduced in Vukan (2005) and similarly in Ceder (2007), who discusses service adjustments and improvements as well as fare collection and fare structure as demand factors. Yet, in both textbook a mathematical formulation of these interactions is not described. To some degree closer to our objectives is the work of Liu et al. (2016) who looked into these relationships in a bilevel model, reflecting the operators and individuals’ decisions. They discussed the fare strategy for a bus-subway corridor and focused on whether transfers are included in the trips. They also did not aim to show a general network model.

One reason for the limited literature on general network performance formulations regarding the triangular relationships “fare – demand – service quality” we suggest is that
the assumption that fare revenue increases immediately transform into service improvements is not realistic. The customers are only, if ever, likely to experience the benefit of their increased payments in the longer run.

Fig. 1. City divided into centre and periphery with different PT network structure

Nevertheless, assuming this ideal case, we aim to address this shortcoming by creating a
complete model reflecting these interactions. Our methodology is defined based on the model proposed by Daganzo (2010), who discussed the interaction between three parameters: network structure, distance between stops and headway. Given a social welfare function considering both operator and travel costs, he optimized these three parameters. In his paper, the city was assumed to have a hybrid structure, consisting of two major parts, a center where the services operate on a grid network and a periphery with hub and spoke service as illustrated in Fig. 1, where the shadowed area would be the grid and the blank periphery the area operated with hub and spoke services. Fig.2 shows the quarters of a squared city networks, hub and spoke in a), grid in b), and the hybrid network in c).

3. Methodology

3.1. Model Overview
Daganzo’s main interest was the understanding of what type of service should be operated for a certain, fixed demand. Thus, his model did not consider demand elasticity due to changes in disutility, nor did it include fare in the model. In contrast we consider the summation of the disutility of a fixed population, including those who might not be using public transport if it becomes too unattractive or expensive. We focus on one PT mode, which we generally consider to be bus (though our methodology would not change if it represents a rail-based service instead) and consider an additional alternative mode which passengers would choose instead in case it is more attractive. We introduce this mode as a potential proxy and limiting case for “rejected journeys”. Taxi is considered as an alternative mode and use parameters reflecting taxi costs; though depending on factors such as car ownership rates, distance and slow mode facilities obviously alternatives would be chosen by many travelers. We argue that our choice of taxi might be seen as conservative estimate and a lower limit for demand elasticity. That is, if taxi is more attractive than the public transport service provided, then clearly the public transport option is unattractive.

We aim to find the flat fare that minimizes the total social disutility of a PT operator and the total travelling population. This is expressed as

$$z = \lambda (\bar{z}_o + \bar{z}_p) + (\hat{\lambda} - \lambda)(\bar{z}_x + \frac{f_x}{\mu})$$

(1)

where $z$ denotes the total social disutility for the whole population $\lambda^{total}$ of concern. $\bar{z}_o$
stands for the disutility of the public transport operator per passenger, while $\tilde{z}_p$ stands for the disutility of a public transport user. Both $\tilde{z}_o$ and $\tilde{z}_p$ are obtained in line with Daganzo’s model. $\tilde{z}_x$ is the disutility of those who chose to use the alternative mode, taxi. We separate in the above formulation the public transport fare term from the total operator and passenger disutility as this is our main variable of concern. $\lambda$ denotes the actual demand for public transportation, and $f_x$, the taxi fare; $\mu$ is the time value required to convert all disutilities, including operator costs, into a single time equivalent unit.

The disutilities are obtained by following equations for public transportation operators and users respectively:

$$\tilde{z}_o = \pi_v V + \pi_M M + \pi_L L \quad \text{(2)}$$

$$\tilde{z}_p = A + W + T + \delta / v_w e_T \quad \text{(3)}$$

Both Eqs (2) and (3) are based on Daganzo’s model. Eq. (2) stands for the disutility of operators per PT passenger, derived by the sum of fixed cost and operational cost. $L$ is the summation of the infrastructure length in the periphery and in the center, to approximate fixed capital costs; $V$ stands for the sum of the vehicle distance and $M$ for vehicle hours traveled during rush hours, thus both reflecting operational costs. $L$, $M$ and $V$ are converted and weighted into monetary costs through the three parameters $\pi_v, \pi_M$ and $\pi_L$. Eq. (3) similarly adds up the walking access time ($A$), waiting time ($W$), travelling time ($T$) and transfer penalties ($\delta / v_w e_T$), as the disutility of each PT passengers. $\delta$ stands for the weight of transfer time, while $v_w$ indicates walking speed and $e_T$ the expected number of transfers.

$$\tilde{z}_x = \frac{E_x}{v_x} \quad \text{(4)}$$

Eq. (4) obtains the disutility of our newly introduced taxi mode as distance traveled divided by velocity. It is assumed that the proxy disutility does not include walking access/egress time, transfer penalties nor waiting time. Ignoring the former two seems realistic for a taxi type service. Average waiting time until a taxi arrives could be added as fixed term in the utility function though for simplicity we omit it. Furthermore, since all costs are included in a single disutility function average waiting would not be distinguishable from a fixed fare that is part of $f_x$ as we discuss later. To reflect that the attractiveness (and availability) of taxi will reduce with more demand we consider its speed $v_x$ to be a function of demand.

We assume a uniform demand distribution where all OD pairs are equally likely which is clearly not realistic but can be considered the worst case for public transport as Daganzo
also notes. Any more concentrated demand will make it easier for public transport to focus its service and attract more demand. Therefore, also regarding this assumption, our model can be considered as a “lower limiting case”. Given these assumptions, optimization is done with respect to four decision variables: $\alpha$ denoting the proportion of the square city center with a grid PT network as in Figures 1 and 2, $s$ determining the grid size and with it the distance between stops, $H$ for headway as well as fare $f$. The input parameters that are utilized in above equations are $D$ denoting the length of the square city, $v_c$ denoting commercial speed of vehicles, $\tau$, the time lost per stop due to the required door operation, deceleration and acceleration; and the time added per boarding passenger, $\tau'$ (hr/pas). (If the effect of alighting is significant, it can be usually subsumed into $\tau'$.) In users’ disutility calculation, $v_w$ represents the walking access speed.

All of the remaining model parameters except for demand $\lambda$ are determined as below following Daganzo. Eq. 5 estimates the total distance operated as a function of city size as well as decision values $\alpha$ and $s$. For the determination of vehicles operated as in (6) further the third decision variable, headway, is used. Eq. 7 provides how the peak hour velocity is obtained while Eq. 8 peak hour traveled time is formulated. Formulation of Eqs. 9 to 12 are also from Daganzo’s paper for passengers’ disutility.

For the model we further need to obtain the proportion of passengers that travel within the city center, within the periphery and those travelling within both parts of the city in order to obtain the expected distance travelled in the network. Since Daganzo’s split of the travelling groups does not appear fitting to us for our research purposes, we derive all of the expected values for vehicular travelling distance ($E$) in the appendix.

\[
L = \frac{D^2(1 + \alpha^2)}{s} \quad (5)
\]
\[
V = \frac{2D^2(3\alpha - \alpha^2)}{sH} \quad (6)
\]
\[
\frac{1}{v_c} = \frac{1}{v} + \frac{\tau}{s} + \frac{2.5(1 + e_T)\tau'\lambda H}{(3\alpha - \alpha^2)D^2} \quad (7)
\]
\[
M = \frac{V}{v_c} \quad (8)
\]
\[
e_T = 1 + \frac{1}{2}(1 - \alpha^2)^2 \quad (9)
\]
\[
A = \frac{s}{v_w}
\]
\[
W = \left[\frac{2 + \alpha^3}{3\alpha} + \frac{(1 - \alpha^2)^2}{4}\right]H
\]
\[
T = \frac{E}{v_c}
\]
\[
E_{pt} = \frac{D(12 - 9\alpha - 9\alpha^2 + 23\alpha^3 - 5\alpha^4 - 5\alpha^5 + 2\alpha^6 + \alpha^7)}{12}
\]

3.2. Taxi parameter settings
For the alternative mode, the velocity is calculated as in Eq. (14). The coefficients are based on a free flow case with velocity of 40km/h; and a “current” or base case, where the velocity is 25 km/h and the flow is the original demand of private vehicle transportation. We also assume when there is no PT demand, that is all potential PT users utilize instead taxi, the velocity of taxi would be 5 km/h, while when half of current PT demand are moved to taxi, the velocity would be 10km/h. Denoting PT demand by \(\lambda\), current PT demand as \(\lambda^0\) and taxi speed as \(v_x\) we then utilise the following four \((\lambda, v_x)\) points: \((\hat{\lambda}, 40\), \((\lambda^0, 25\), \((\lambda^0/2, 10)\) and \((0, 5)\). With these we obtain a function \(v_x(\lambda)\) through curve fitting. We further obtain taxi fares \(f_x\) as combination of base fare \(f_{xb}\) and distance depending fare \(f_{xd}\) as in (14):

\[
f_x = f_{xb} + f_{xd}E_x
\]

3.3. Demand Elasticity Estimation
Introducing an alternative mode allows us to consider demand sensitivity. The public transport demand \(\lambda\) is obtained as a function of the relative disutility of public transport and taxi. We test both linear as well as logit formulations.

For the linear model, Figure 3 illustrates the PT demand as a function of taxi disutility and taxi fare. Given the original fare and disutility, we assume that all demand \(\lambda^{total}\) is utilizing public transport. We set this demand as upper limit, so that we also refer to it as “potential public transport users”.

A given base case is marked with 2 in the figure. When the sum of public transport fare and disutility drops below a lower threshold, we consider that there is a upper bound for PT demand as flat part of the curve marked with 1. Further, when the sum of fare and travel disutility equals to that of PT, half of the passengers would turn to taxi, indicated
by point 3.

\[
\lambda = \begin{cases} 
\lambda^0 \left( 1 - \frac{z_p + \frac{f_x}{\mu} - \frac{z_0}{\mu} - \frac{f_0}{\mu}}{2(z_x + \frac{f_x}{\mu} - \frac{z_0}{\mu} - \frac{f_0}{\mu})} \right) & \text{if} \ 2 - \frac{2\lambda}{\lambda^0} \geq \frac{z_p + \frac{f_x}{\mu} - \frac{z_0}{\mu} - \frac{f_0}{\mu}}{z_x + \frac{f_x}{\mu} - \frac{z_0}{\mu} - \frac{f_0}{\mu}} \\
\tilde{\lambda} & \text{if} \ 2 - \frac{2\tilde{\lambda}}{\lambda^0} \geq \frac{z_p + \frac{f_x}{\mu} - \frac{z_0}{\mu} - \frac{f_0}{\mu}}{z_x + \frac{f_x}{\mu} - \frac{z_0}{\mu} - \frac{f_0}{\mu}} \end{cases}
\]

3.4. Fares/Operation cost as Evaluation Indicator

In addition to service quality and travel disutility as another indicator to evaluate specific fare scenarios we consider the percentage of operational costs (including fixed infrastructure costs) covered by the fare revenue as in Eq. (16). The indicator helps us to express the operator dilemma of, on the one hand, aiming to provide a good (and expensive) service, but, on the other hand, financial constraints that require the operator to not overextend their service if the demand is too low. In the following scenarios \( \phi \) will further help us to generally show for which type of cities larger subsidy requirements can be expected.
\[ \phi = \frac{f_{PT}}{z_o} \]  

(16)

4. Results

4.1. Base Case Settings

In the following we will investigate a range of scenarios. To allow comparison with Daganzo’s work we use Barcelona again as reference case. The 2013 modal share in Barcelona is 12.28% for bus (Pan, 2013) and 2% for taxis (Barcelona City Council; 2014). This leads us to an estimate of \( \lambda \approx 22854 \text{ pas/h} \). Following Daganzo’s \( \lambda^0 = 20000 \text{ pas/h} \) with a speed of \( v_x^0 = 25 \text{ km/h} \) we then obtain the taxi speed function through curve fitting with the four \((\lambda, v_x)\) as explained in previous section. We use a polynomial of third order leading to Eq. (17):

\[ v_x = 2.28 \times 10^{-6} \lambda^3 - 9.58 \times 10^{-4} \lambda^2 + 1.26 \times 10^{-1} \lambda + 5 \]  

(17)

For the base case we use a public transport fare \( f^0 = $2.29 \) which reflects the current flat fare for buses in Barcelona. Taxis have a base fare of $7.46 and a distance depending surcharge of $1.28/km, also in line with the current taxi fares in Barcelona converted into $ from Tourist Guide Barcelona page (2016).

Table 1 lists the additional input parameter values chosen for the base case. We take these again from Daganzo’s work and refer for further discussion on their appropriateness to his paper.

<table>
<thead>
<tr>
<th>Input Variable</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>S</td>
<td>30</td>
</tr>
<tr>
<td>( \tau' )</td>
<td>s/pas</td>
<td>1</td>
</tr>
<tr>
<td>V</td>
<td>km-h</td>
<td>25</td>
</tr>
<tr>
<td>( \delta )</td>
<td>Km</td>
<td>0.03</td>
</tr>
<tr>
<td>$L</td>
<td>$/km-h</td>
<td>9</td>
</tr>
<tr>
<td>$V</td>
<td>$/veh-km</td>
<td>1</td>
</tr>
<tr>
<td>$M</td>
<td>$/veh-h</td>
<td>30</td>
</tr>
<tr>
<td>D</td>
<td>Km</td>
<td>10</td>
</tr>
<tr>
<td>( \lambda^0 )</td>
<td>pas/h</td>
<td>20,000</td>
</tr>
</tbody>
</table>
### 4.2. Comparison for different fare setting in base scenario

Firstly, to obtain some general understanding of how the service changes if fares are considered, we compare Daganzo’s scenario with the optimal fare scenario. The results are shown in Table 2.

<table>
<thead>
<tr>
<th>Decision variables and resulting costs for realistic optimal fare and for Daganzo’s model with fares added</th>
<th>Daganzo (2010), with current fares added</th>
<th>Optimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.65</td>
<td>0.65</td>
</tr>
<tr>
<td>s [km]</td>
<td>0.45</td>
<td>0.44</td>
</tr>
<tr>
<td>H [min]</td>
<td>4.17</td>
<td>3.90</td>
</tr>
<tr>
<td>( \bar{z}_o ) [min]</td>
<td>6.95</td>
<td>6.63</td>
</tr>
<tr>
<td>( \bar{z}_y ) [min]</td>
<td>42.54</td>
<td>42.14</td>
</tr>
<tr>
<td>( \bar{z}_x ) [min]</td>
<td>16.00</td>
<td>10.66</td>
</tr>
<tr>
<td>z [min]</td>
<td>62468</td>
<td>18577</td>
</tr>
<tr>
<td>fare[$]</td>
<td>2.29</td>
<td>1.47</td>
</tr>
<tr>
<td>( \lambda ) [pax/min]</td>
<td>333.33</td>
<td>380.95</td>
</tr>
<tr>
<td>( \phi )</td>
<td>0.99</td>
<td>0.67</td>
</tr>
</tbody>
</table>

We observe that the social welfare maximising fare is lower than the fare in the current situation. Furthermore, the headway is lower. This creates an increase in public transport passengers, in fact all prospective passengers, i.e. \( \lambda = \hat{\lambda} \) and lowers their disutility. The increase in demand also explains why the operational cost per passenger remains nearly constant, in fact is even slightly reduced despite the better service in terms of frequency. We observe though that the cost coverage \( \phi \) reduces significantly from near cost coverage with the original fare to only 2/3 of the operational costs being covered in the social welfare optimal case, illustrating the conflict from a transport planning perspective. Obviously if we introduce weights for operator costs and sum of passenger disutility we might obtain different solutions.

### 4.3. Changes in the minimum fare

To explore the conflict in cost coverage versus social welfare further, in the following we vary the minimum charge the public transport operator is willing to accept and find the social welfare optimum which might or might not be at this lower fare level.
For the discussion we require a range of figures: Fig. 4 shows the changes in the three decision variables $H, s$ and $\alpha$ that determine the service quality. We plot the scenario value divided by its respective value at the original fare level ($2.29$) in order to better illustrate the general tendency. We remind that an increase in $\alpha$, decrease in $s$ or $H$ improves the service quality, and vice versa. Fig. 5 illustrates the PT demand elasticity with respect to fare and Fig. 6 illustrates the travel disutility for the operator, PT passengers and taxi customers. Fig. 7 shows the objective function value changes. Finally Fig. 8 is the value of the cost coverage ratio $\phi$.

We observe that as the lower fare boundary increases, the resulting demand and passenger disutility stays constant for fares up to approximately $2.3$ (Figs. 5 and Fig. 6). The operator’s disutility is in contrast increasing significantly in the range of $1.65$ to $2.3$ as in Figure 6. We can conclude from this that, from a customer perspective, a moderate fare increase can be compensated by better service quality as operators succeed in attracting passengers despite fare increases. A higher fare than $2.3$, will lead to a demand decrease though (Fig. 5).

Therefore $2.3$ must be seen as a critical point. When the fare exceeds this, not only does the demand decrease but also the service starts to get worse as shown in Figs. 4 and 5. To note is that the value of $\alpha$ remains constant, especially after the critical point, while the values of $s$ and $H$ increase, though gradually, making the service worse. $2.3$ is also an inflection point for the total, operators’, users’ and taxi disutility as Figs. 6 and 7 illustrate. Looking at the public transport demand in more detail, one can observe that the curve is concave with respect to fare at the critical point, i.e. marginal increases at a fare around $2.3$ have a larger effect than fare increases at a higher level. This can be explained by the decrease in competitiveness of taxi when there are increasing numbers of customers and vehicles on the road, thus when more people turn to taxi, its disutility would increase.

Fig. 8 shows the changes in the value of $\phi$ still with respect to the lower fare boundary. The two critical points, i.e. social welfare optimal fare $1.46$ and demand sensitivity point at $2.3$, stand out here as well as inflection points. In between these points is a parabolic curve, with a peak at $1.85$. When fare gets higher than this, before the demand starts to drop, the coverage gets lower. Taking these observations all together we suggest that a fare between $1.47$-$1.85$ is reasonable and Pareto optimal. Higher fares can remain or improve the service level but are negative from a disutility perspective for operators, higher fares than $2.3$ are also unfavorable for customers.
Fig. 4 Decision variable changes in base scenario

Fig. 5 PT Demand changes in base scenario
Fig. 6 Travel disutility changes in base scenario with fare (Absolute value)

Fig. 7 Total disutility changes in base scenario (Absolute value)

Fig. 8 Cost Coverage Ratio $\phi$ and fare changes in base scenario
4.4. Sensitivity to city characteristics

The following Figures 9 and 10 represent the decision variables in optimum cases, when either total demand $\lambda_{PT0}$ or city size $D$ varies while the other input parameters remain constant. On the x-axis is the multiplier of the variable compared to the base scenario. A decrease in $\lambda_{PT0}$ and an increase in $D$ both correspond to a decrease in demand density so that opposing trends in the figures can be observed.

When the demand density increases, both $s$ and $H$ reduce, since more demand causes the operators to provide a better service. We observe that in contrast the value of grid versus hub- and spoke network, $\alpha$, remains almost unchanged, similar to Fig. 4 where $\alpha$ also remains fairly unchanged for fare alternations.
When the demand density decreases significantly, people would tend to prefer the alternative mode, as the public transport operator can not maintain a good service. Our model suggests that a zero fare is optimal, but this is still not sufficient to encourage public transport usage. (Our model suggests that if the positivity constraint for fares are omitted, from a social welfare perspective, a negative fare would be optimal).

4.5. Sensitivities regarding taxi fare ($f_x$)

As argued we model the alternative mode to public transport as “taxi” though consider this more general as a proxy for alternative modes that one might take as it is the generalized cost consisting of (fast) travel time and (high) fare for taxi that determines the modal split. Considering the range of alternatives and variations in value of time among the population, we vary the taxi fare to understand again social welfare and public transport implications if competing modes become more or less attractive. Figure 11 illustrates the resulting optimal public transport service characteristics.

In general, in extreme cases when a large proportion of demand chooses not to utilize PT, the higher the fare of the alternative mode, the less pressure this puts on both the quality of service and the fare of PT.

As fare $f_x$ increases, within a certain range where most demand would be using PT, the optimum PT fare would increase as well and the additional revenues are invested into headway reductions, so that the PT service quality improves but overall disutility remains largely unchanged. However, when the taxi fare exceeds a certain level, in our case study between 0.7-0.8 times of the base level from Table 1, further increases in $f_x$ would trigger reductions in PT fare, while the disutility of operators keeps the same, as $s$ and $H$ remains stable. This can be interpreted via the objective function we took. We were trying to obtain the social optimum, thus here as taxi fare increases, in return, a “discount” for PT fare is expected, until this leads to the “free ride” situation, where fluctuations in service quality occur, generally increasing, to compensate the increasing fare in taxi again.
Fig. 11. Optimum when taxi fare varies

Since taxi fleet is also an input variable in our model, we confirmed the sensitivities regarding it, yet the result remain the same, as $\lambda_{PT}$ remains at the maximum at the optimum scenario, unaffected by the taxi fleet.

5. Discussion and Conclusions

In this paper we advance the continuous approximation of Daganzo (2010) to illustrate the effect of fare changes on public transport quality, public transport usage and overall social welfare including operator disutility, income and passenger disutility. In contrast to Daganzo’s work we therefore include fare and mode choice, in our case represented by a choice between public transport and taxi. Clearly the goal of this work has not been to derive detailed advice for specific cities but to describe general tendencies. We suggest following observations grouped into three categories are noteworthy:

5.1. (Optimal) Fare Policy

First of all, there exists an optimum fare that minimizes the overall disutility, i.e. maximizes the social welfare. It is quite low, compared with the current fare in our reference case that is based on Barcelona data. At this optimal fare point all potential demand would use PT instead of the alternative mode. Secondly, there is an inflection point that we could identify when we increase the fare from its optimal value. For slight increases, the additional fare revenues can be utilized to keep attracting people to public transport by offering better public transport. However, once the inflection point is crossed, this further compensations are not attractive anymore and passengers start to choose the alternative mode. Thirdly, we can identify a point in between the optimal fare and the
inflection point that maximize the revenue of the public transport operator considering fare income and expenses to provide the service. Taken all these observations together we suggest that fares in between the social welfare optimizing point and the inflection point can all be considered as Pareto optimal. We note that, interestingly, the current actual bus fare in Barcelona is very close to the revenue maximizing point though do not want to read too much into this due to the many simplifications made in our model.

5.2. PT service characteristics
We find that the optimal PT network structure, here represented by the split between grid and radial network, does not vary much within a given city size, since no matter how the other variables change, we observe that \( \alpha \) hardly changes at all. \( \alpha \) appears to be mainly influenced by the mode characteristics, as described in Daganzo’s paper, which is, however, not the focus of this paper. Instead, the values of \( s \) and \( H \), station spacing and bus headway respectively, vary a lot depending on city characteristics, PT fare and alternative mode fare. In general we find that as the city gets larger, subsequently both \( s \) and \( H \) increase vice versa.

5.3. Implications for modes competing with PT.
Finally, in this section we change perspective and consider the implications of our study for the provider of alternative modes such as conventional taxi or ride-sourcing which in general we expect to be more convenient (faster) but also more expensive than mainstream fixed public transport.

On the one hand, we observe that the taxi fleet, given PT is convenient enough that most people could choose to use PT, has little influence on the system. On the other hand, all decisions variables (except \( \alpha \)) are sensitive to the changes in the taxi fare. We observe that the optimal PT fare in general shows a fairly negative linear relationship with a medium (realistic) range of taxi fares. This is because when taxi fare is rather high, more demand would head to PT, and overall disutility tends to become higher. Hence to relief the passengers’ burden on travel, while providing the same service quality, PT fare is comparatively low, and the opposite scenarios follow the same logic. Obviously, when taxi fare is too high people would not take taxi anyway, and when it is too cheap all demand would prefer taxi.

On the one hand, when the taxi fare declines at a low level, the optimal PT fare turns out to be dropping as well. This is intuitive as given a taxi fare being very low a significant
amount of demand might prefer taxi instead of PT. Thus to compete, the PT operator should decrease its fare to attract more demand. If the “price war” continues it results in in the PT operators’ disability to keep up the quality of its service, reflected by the growth in $s$ and $H$. On the other hand, when the taxi fare is very high, the PT fare tends also to the lower boundary set by the regulator, in our example zero. Further, the service quality increases, by dropping $s$ and $H$, compensating overall passengers’ disutility in another way. These example show the complex interaction between the PT and alternative mode fare. Further, from a social welfare perspective we can conclude that the existence of a reasonable (but not too cheap) alternative to public transport can have positive effects. Above results also suggest that the strategy that the taxi and PT operators may take in their fare settings might have to be regulated from a social welfare perspective.

Our model is simple and meant to provide general insights. In further work we suggest applications to specifically test the effects of new modes such as Uber, car sharing, etc. could be evaluated. We suggest our model could also be used as a building block within larger models that aim to develop general guidelines on the interaction between land-use, pricing and PT network design. Clearly the model can be further advanced in a number of directions such as aiming to model various kinds of dynamics as well as non-uniform demand patterns.

References
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APPENDIX A Derivation Of Public Transport Distance $E_{PT}$
First for $E_{PT}$ we consider the distance travelled in the periphery and in the center separately. $E_{PT}$ in the periphery, we mark as $R_P$. Consider the distance between a random point on the boundary and a random point in the periphery on the same quadrant.
Distance of a random point from cordon with side $n$ on the periphery to a random point in the boundary is $\frac{3}{4}(n-d)$, and the PDF of it is $\frac{2n}{D^2}$. Thus, $E(R'_P) = \frac{D}{4}(2 - 3\alpha + \alpha^3)$. For C-P or P-C case, there is $E(R_P|C-P,P-C) = E(R'_P)$. For the P-P case, it becomes $E(R_P|P-P) = 2E(R'_P)$. Thus, with one PDF of one of the O or D in the periphery already considered with $\frac{2n}{D^2}$, we only need to multiply the possibility of the other end, as $2\alpha^2(1 - \alpha^2)$ for the C-P or P-C case and $(1 - \alpha^2)^2$ for P-P case when we are calculating the overall $E(R_P)$, as:

$E(R_P) = \frac{D}{2}(1 - \alpha^2)(2 - 3\alpha + \alpha^3)$

Similarly marking the distance travelling in the center by PT as $R_C$, we divide 3 cases for calculation, as: a) both O and D are in the center with probability of $\alpha^4$, b) Both O and D are in the periphery with probability of $(1 - \alpha^2)^2$, and c) either O or D is in the center and the other in
the periphery with probability of $2(1 - \alpha^2)\alpha^2$.

For case A, it is clear that:

$$E(R_C|A) = \frac{2}{3}d = \frac{2}{3}\alpha D$$

For case B, three cases are separated again, as a) Both OD are at the middle point with probability of $(1 - \alpha)^2$; b) Neither OD are at the middle point with probability of $\alpha^2$; and c) Only one of the OD are at the middle point with probability of $2\alpha(1 - \alpha)$. Determining the expected values in each case and then adding them up, we obtain:

$$E(R_C|B) = E(R_C|Ba) \cdot (1 - \alpha)^2 + E(R_C|Bb) \cdot \alpha^2 + E(R_C|Bb) \cdot 2\alpha(1 - \alpha)$$

$$= \frac{7}{8} d \cdot 2\alpha(1 - \alpha) + \frac{3}{4} d (1 - \alpha)^2 + \frac{11}{12} d \alpha^2 = \frac{\alpha D}{12} (9 + 3\alpha - \alpha^2)$$

For case C, three cases are separated again, as a) The points are on the same side with probability of $1/4$; b) The points are on opposite sides with probability of $1/4$; c) the points are on adjacent sides with probability of $1/2$. Again by deriving them separately and summing up, it follows:

$$E(R_C|C) = E(R_C|Ca) \cdot \frac{1}{4} + E(R_C|Cb) \cdot \frac{1}{4} + E(R_C|Cc) \cdot \frac{1}{2} = \alpha \left( \frac{\alpha}{12} + \frac{3}{4} \right) D$$

Thus, for PT travels, the value of $E_{PT}$ is the summation of $R_C$ and $R_P$ of each cases, being:

$$E_{PT}^C = \frac{D}{12} (6 + 3\alpha^2 + 2\alpha^3 - 5\alpha^4 + \alpha^5 + 2\alpha^6 + \alpha^7)$$

$$E_{PT}^P = \frac{2}{3} \alpha D$$

$$E_{PT}^P = \frac{D}{12} (12 - 9\alpha + 3\alpha^2 + 5\alpha^3)$$

And the total would be:

$$E_{PT} = E(R_P) + E(R_C) = \frac{D(12 - 9\alpha - 9\alpha^2 + 23\alpha^3 - 5\alpha^4 - 5\alpha^5 + 2\alpha^6 + \alpha^7)}{12}$$

**APPENDIX B**  Derivation of Alternative Mode Distance $E_x$

We assume that taxi is a much more flexible mode of transportation here, where the fare depends on distance while they can pick up customers at any blocks, where we assume the boarding and off boarding spots overlaps with the stations. Thus, we can consider the whole taxi network as a grid network similar to the bus network within the city center so that the expected travelling distance, $E_x$, equals to $\frac{2}{3} D$. 